**Mathematical Tripos Part II** Michaelmas 2015
**Integrable Systems, hand out 3: evolution of scattering data** Dr A.C.L. Ashton

In §3.1 we discussed how to construct the scattering data

$$S = \Big\{ \{\chi_n, c_n\}_{n=1}^N, R(k) \Big\}$$

for a Schrödinger operator $L$ associated with a potential function $u = u(x)$. Now we do the clever bit: we allow the potential $u$ to evolve according to the KdV equation and look at the scattering data for the corresponding Schrödinger operator $L$, which now depends on time. Throughout we refer to $A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)$ and use the fact that because $u$ satisfies KdV, then $L_t = [L, A]$. We will also make use of the fact that $A = 4\partial_x^3$ for large $|x|$, since we've assumed $u$ vanishes for $|x|$ sufficiently big.

### 1. Evolution of continuous spectrum

Using the same argument as in §3.1.1 we can still show that for each $\lambda = k^2 > 0$ there is a $\varphi$ such that $L\varphi = k^2\varphi$ and

$$\varphi(x,k,t) = \begin{cases} e^{-\mathrm{i}kx} & \text{as } x \to -\infty, \\ a(k,t)e^{-\mathrm{i}kx} + b(k,t)e^{\mathrm{i}kx} & \text{as } x \to +\infty, \end{cases}$$

but now the functions $a$ and $b$ change with $t$ (because $u$ does). For fixed $k$, we can differentiate the equation $L\varphi = k^2\varphi$ with respect to $t$, as we did in the proof to the Isospectral flow theorem. Using the fact that $L_t = [L, A]$ we find

$$0 = (L - k^2)(\varphi_t + A\varphi).$$

So the function $\varphi' = \varphi_t + A\varphi$ also satisfies $L\varphi' = k^2\varphi'$. Also, since we assume that $u \equiv 0$ for $|x|$ sufficiently large (so $A \equiv 4\partial_x^3$ for $|x|$ sufficiently large) we have the asymptotic behaviour

$$\varphi'(x,k,t) = \begin{cases} 4\mathrm{i}k^3 e^{-\mathrm{i}kx} & \text{as } x \to -\infty, \\ (a_t + 4\mathrm{i}k^3 a)e^{-\mathrm{i}kx} + (b_t - 4\mathrm{i}k^3 b)e^{\mathrm{i}kx} & \text{as } x \to +\infty. \end{cases}$$

Now $\varphi$ and $\varphi'$ satisfy the same eigenvalue problem, so we know that[1] the Wronskian, $W(\varphi', \varphi)$, is constant. Computing it for large, negative $x$ we deduce that $W(\varphi', \varphi) = 0$ so that $\varphi' = C(t)\varphi$. Looking at the large negative $x$ behaviour of both $\varphi'$ and $\varphi$ we deduce that $C(t) = 4\mathrm{i}k^3$. So $\varphi' = 4\mathrm{i}k^3\varphi$ and by comparing the coefficients of $e^{\pm\mathrm{i}kx}$ in the large $x$-behaviour of these two functions we arrive at

$$a_t + 4\mathrm{i}k^3 a = 4\mathrm{i}k^3 a, \quad b_t - 4\mathrm{i}k^3 b = 4\mathrm{i}k^3 b,$$

which we can integrate up immediately, $a(k,t) = a(k,0)$ and $b(k,t) = b(k,0)e^{8\mathrm{i}k^3 t}$. The transmission and reflection coefficients become

$$T(k,t) = T(k,0), \quad R(k,t) = R(k,0)e^{8\mathrm{i}k^3 t}.$$

You should stand back and stare in awe at this result. The potential $u = u(x,t)$ evolves according to a complicated, third order, nonlinear PDE, but the scattering data from the continuous spectrum of the associated Schrödinger operator evolve in a super-simple way. This is *amazing*!

### 2. Evolution of discrete spectrum

For each $t$ we can still construct our bound states $\{\psi_n\}$ and by the isospectral theorem the discrete eigenvalues $\{\chi_1, \ldots, \chi_N\}$ are independent of time. The bound states are then characterised by

$$\psi_n(x,t) = c_n(t)e^{-\chi_n x} \quad \text{as } x \to \infty.$$

As before, consider the new functions $\psi_n' = \partial_t\psi_n + A\psi_n$. Then by the isopectral theorem $L\psi_n' = -\chi_n^2\psi_n'$ and

$$\psi_n'(x,t) = (\dot{c}_n - 4\chi_n^3 c_n)e^{-\chi_n x} \quad \text{as } x \to \infty$$

[1] See example sheet 2, question 2: if $L\varphi_1 = k^2\varphi_1$ and $L\varphi_2 = k^2\varphi_2$ then the Wronskian $W(\varphi_1, \varphi_2)$ is constant.

since $u$ vanishes for large $x$. Again we look at the Wronskian: since $\psi_n$ and $\psi_n'$ satisfy the same eigenvalue equation we know the Wronskian is constant, and since both functions decay exponentially we find $W(\psi_n', \psi_n) = 0$ so that $\psi_n' = C(t)\psi_n$. However, since $A$ is an anti-symmetric operator[2] and $\|\psi_n\| = 1$:

$$C(t) = C(t)\langle\psi_n, \psi_n\rangle = \langle\psi_n', \psi_n\rangle = \langle\partial_t\psi_n, \psi_n\rangle + \langle A\psi_n, \psi_n\rangle = \tfrac{1}{2}\partial_t\|\psi_n\|^2 = 0$$

so $\psi_n' = \partial_t\psi_n + A\psi_n = 0$. By looking at the large $x$-behaviour of $\psi_n'$ we deduce:

$$\dot{c}_n - 4\chi_n^3 c_n = 0$$

for the normalisation constant $c_n$. Hence

$$c_n(t) = c_n(0)e^{4\chi_n^3 t}.$$

Once again, even though the potential $u = u(x,t)$ evolves according to a complicated nonlinear PDE, the associated normalisation constants evolve very simply!

**Summary of evolution of scattering data**

Suppose $u = u(x,t)$ solves the KdV equation. Then the scattering data for the associated Schrödinger operator $L$ evolves according to

$$S(t) = \left\{ \{\chi_n, c_n(0)e^{4\chi_n^3 t}\}_{n=1}^N, R(k,0)e^{8\mathrm{i}k^3 t}, T(k,0) \right\}.$$

Again we reconstruct the potential $u = u(x,t)$ by solving the GLM equation, by treating $t$ as a parameter for the functions $F$ and $K$. This is the inverse scattering transform, as summarised in Figure 1.

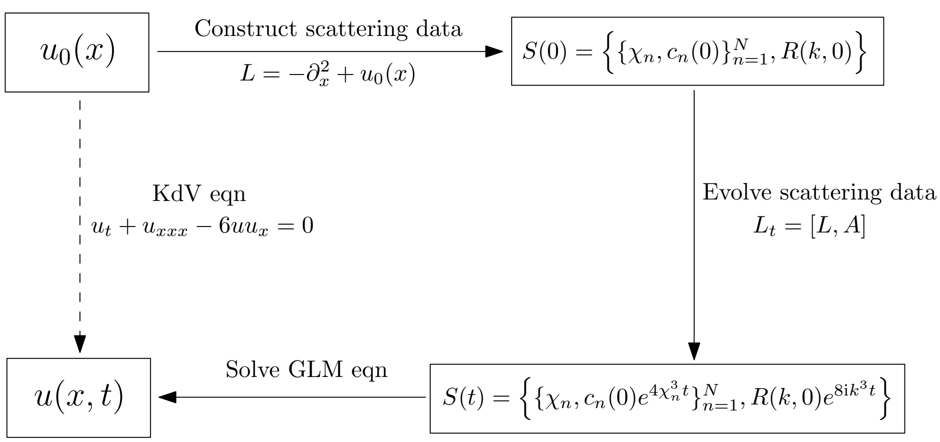


Figure 1: The inverse scattering transform.

[2] See example sheet 2, question 5.