Mathematical Tripos Part II Michaelmas 2015
Integrable Systems, hand out 4: From Lax pairs to zero-curvature Dr A.C.L. Ashton

Recall that a Lax pair of operators (L, A) consisted of a self-adjoint linear operator L and a linear operator A such that

$$L_t = [L, A].$$

The isospectral theorem showed us that the discrete eigenvalues of L are time independent if this equation holds. An alternative approach is to consider the linear problem

$$\begin{cases} L\psi = \lambda\psi, \\ \psi_t + A\psi = 0, \end{cases}$$

and *insist* that the discrete eigenvalues λ are time independent. By differentiating the first equation and using the second to replace ψ_t with $-A\psi$ we find that $\lambda_t = 0$ if

$$L_t = [L, A].$$

That is to say, the Lax equation is the *compatibility condition* between two linear problems. We will show that this approach gives rise to a zero curvature equation.

Let us now concentrate on the linear problem

$$\begin{cases} L\psi = \lambda\psi, \\ \psi_t + A\psi = 0, \end{cases}$$

where for simplicity we'll assume that L and A are scalar differential operators. These equations imply $L_t = [L, A]$ if we enforce $\lambda_t = 0$. Write L and A in the general form

$$L = \partial_x^n + \sum_{j=0}^{n-1} u_j(x,t)\partial_x^j, \quad A = \partial_x^m + \sum_{j=0}^{m-1} v_j(x,t)\partial_x^j.$$

The Lax equation $L_t = [L, A]$ will give rise to a nonlinear PDE for the functions

$$(\mathbf{u}, \mathbf{v}) = (u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1}).$$

We make an important observation: the first equation $L\psi = \lambda\psi$ implies that all derivatives of ψ of order $\geq n$ can be expressed in terms of a linear combination derivatives of order $< n$. This follows by a simple induction argument: we claim $\partial_x^{n+k}\psi$ is a linear combination of $\{\psi, \dots, \partial_x^{n-1}\psi\}$ for $k \geq 0$. It is certainly true for $k = 0$ since rearranging $L\psi = \lambda\psi$ gives us

$$\partial_x^n\psi = \lambda\psi - \sum_{j=0}^{n-1} u_j(x,t)\partial_x^j\psi \equiv \sum_{j=1}^{n} A_j(\mathbf{u}, \lambda)\partial_x^{j-1}\psi$$

and the inductive step follows from this equation and noting $\partial_x^{k+1+n}\psi = \partial_x(\partial_x^{k+n}\psi)$ (if this is not clear you should fill in the details of the induction argument yourself). Let us first write the nth order linear PDE (for ψ) $L\psi = \lambda\psi$ as a first order matrix PDE for $\Psi = (\psi, \partial_x\psi, \dots, \partial_x^{n-1}\psi)^t$

$$\partial_x\Psi = U\Psi, \quad \text{where} \quad U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ \lambda - u_0 & -u_1 & -u_2 & \cdots & -u_{n-1} \end{pmatrix}.$$

Now consider the linear PDE $\psi_t + A\psi = 0$. Differentiating with respect to x $(i-1)$-times

$$(\partial_x^{i-1}\psi)_t + \partial_x^{i-1}\left[\partial_x^m\psi + \sum_{j=0}^{m-1} v_{j-1}(x,t)\partial_x^j\psi\right] = 0, \quad 1 \leq i \leq n.$$

Applying all the derivatives to the square bracket term and using our previous observation, the resulting derivatives of ψ of order $\geq n$ can be expressed in terms of linear combinations of derivatives of order $< n$. The coefficients in this linear combination will depend on λ, $(\mathbf{u}, \mathbf{v})$ and derivatives thereof. Using the fact $\partial_x^{i-1}\psi \equiv \Psi_i$ we find

$$(\Psi_i)_t - \sum_{j=1}^{n} V_{ij}(\mathbf{u}, \mathbf{v}, \lambda)\Psi_j = 0, \quad 1 \leq i \leq n$$

for some appropriate functions $V_{ij} = V_{ij}(\mathbf{u}, \mathbf{v}, \lambda)$. In summary:

$$\begin{cases} \Psi_x = U(\lambda)\Psi, \\ \Psi_t = V(\lambda)\Psi. \end{cases}$$

Since these equations are compatible if the original linear equations were compatible (i.e. $L_t = [L, A]$), we arrive at the zero curvature equations

$$U_t - V_x + [U, V] = 0.$$

Example. Consider $L = -\partial_x^2 + u$ with $u = u(x,t)$ evolving according to KdV. Then $L\psi = \lambda\psi$ can be written in terms of $\Psi = (\psi, \psi_x)$ as

$$(\Psi_1)_x = \Psi_2, \quad (\Psi_2)_x = (u - \lambda)\Psi_1, \quad \text{i.e. } U(\lambda) = \begin{bmatrix} 0 & 1 \\ u - \lambda & 0 \end{bmatrix}.$$

These equations allow us to write derivatives of Ψ of order ≥ 2 as a liner combination of derivatives < 2. We will use this in what follows. Using $A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)$ and $(\Psi_1)_x = \Psi_2$, $\psi_t = -A\psi$ becomes

$$\begin{aligned} (\Psi_1)_t &= -4(\Psi_2)_{xx} + 3u\Psi_2 + 3(u\Psi_1)_x \\ &= -4((u - \lambda)\Psi_1)_x + 3u\Psi_2 + 3u_x\Psi_1 + 3u(\Psi_1)_x \\ &= -4u_x\Psi_1 - 4(u - \lambda)\Psi_2 + 3u\Psi_2 + 3u_x\Psi_1 + 3u\Psi_2 \\ &= (2u + 4\lambda)\Psi_2 - u_x\Psi_1. \end{aligned}$$

Differentiating with respect to x and again replacing $(\Psi_2)_x$ with $(u - \lambda)\Psi_1$ and $(\Psi_1)_x$ with Ψ_2 we find

$$(\Psi_2)_t = \big((2u + 4\lambda)(u - \lambda) - u_{xx}\big)\Psi_1 + u_x\Psi_2.$$

So the appropriate matrices U, V for the zero curvature equations are

$$U(\lambda) = \begin{bmatrix} 0 & 1 \\ u - \lambda & 0 \end{bmatrix}, \quad V(\lambda) = \begin{bmatrix} -u_x & 2u + 4\lambda \\ 2u^2 - u_{xx} + 2u\lambda - 4\lambda^2 & u_x \end{bmatrix}.$$

You should check that $U_t - V_x + [U, V] = 0$ is equivalent to KdV.