Mathematical Tripos Part III Lent 2017
Distributions, hand out 3: fundamental solutions Dr A.C.L. Ashton

We claimed that the locally integrable function

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a fundamental solution to the Heat Operator $P(D) = \partial_t - \Delta_x$ in the coordinates $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. Since E is locally integrable it certainly defines an element of $\mathcal{D}'(\mathbf{R}^{n+1})$. In addition, a routine computation shows that for each $t > 0$

$$\frac{\partial E}{\partial t} - \Delta_x E = 0. \tag{$\star$}$$

According to the definition of the distributional derivative

$$\langle P(D)E, \varphi\rangle = \langle E, P(-D)\varphi\rangle = -\int_0^\infty \left(\int E(x,t)(\varphi_t + \Delta_x\varphi)\,\mathrm{d}x\right)\mathrm{d}t$$

Since E is locally integrable, we can write the latter integral as

$$-\lim_{\epsilon\to 0}\int_\epsilon^\infty \left(\int E(x,t)(\varphi_t + \Delta_x\varphi)\,\mathrm{d}x\right)\mathrm{d}t,$$

where the limit is taken from above. A note for intuition: we did this so we could integrate over a region $t \geq \epsilon > 0$ in which $P(D)E = 0$. On integrating by parts we see that

$$\int_\epsilon^\infty \left(\int E(x,t)(\varphi_t + \Delta_x\varphi)\,\mathrm{d}x\right)\mathrm{d}x = \int_\epsilon^\infty \int \partial_t(E\varphi)\,\mathrm{d}x\,\mathrm{d}t - \int_\epsilon^\infty \left(\int \varphi(\partial_t - \Delta_x E)\,\mathrm{d}x\right)\mathrm{d}t.$$

The latter integral vanishes, from the observation in ($\star$), and the former integral is

$$-\int E(x,\epsilon)\varphi(x,\epsilon)\,\mathrm{d}x$$

by the fundamental theorem of calculus. In summary

$$\langle P(D)E, \varphi\rangle = \lim_{\epsilon\to 0}\int E(x,\epsilon)\varphi(x,\epsilon)\,\mathrm{d}x = \lim_{\epsilon\to 0}\frac{1}{(4\pi\epsilon)^{n/2}}\int e^{-|x|^2/4\epsilon}\varphi(x,\epsilon)\,\mathrm{d}x.$$

On making the substitution $x = 2\sqrt{\epsilon}y$ and applying the dominated convergence theorem, we find $\langle P(D)E, \varphi\rangle = \varphi(0,0)$ for each $\varphi \in \mathcal{D}(\mathbf{R}^{n+1})$. Hence $P(D)E = \delta_0$, i.e. E is a fundamental solution to the Heat operator, as claimed.