On the compound matrices and symmetric matrix functions of a given symmetric matrix

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Abstract

This article studies (i) the k-th compound matrix $C_k(\mathrm{A})$, $\quad k = 0, \ldots, n$, of some symmetric $n \times n$ 'parent' matrix A, (ii) the matrices $\mathrm{A}^{(n,k,s)}$, $\quad s = 0, \ldots, k$, determined by means of $C_k(\mathrm{I} + t\mathrm{A}) = \sum_{s=0}^{k} t^s \mathrm{A}^{(n,k,s)}$, and (iii) and further matrices $\mathrm{M}_k(\mathrm{A}^{s_1}, \ldots, \mathrm{A}^{s_k})$, which are defined by endomorphisms of the space of k-vectors in which $C_k(\mathrm{A})$ acts. The matrices of (iii) are termed monomial symmetric matrices (MSMs), since they are totally symmetric functions of their arguments, here restricted to be powers of the parent matrix A. The matrices of (i) and (ii) are simple special cases thereof. Discussion provided of the properties of such matrices includes detailed study of their eigenvalues (see Propositions 7 and 11) and characteristic equations (CEs), and is illustrated by full treatments of a wide selection of typical examples. A striking formal analogy is described between matrices of types (ii) and (iii) on the one hand, and elementary symmetric and monomial symmetric functions of subsets of k of the eigenvalues of the parent matrix.

Special attention is given to the $s = 1$ case $\Delta_k(\mathrm{A}) = \mathrm{A}^{(n,k,1)}$ of the matrices of type (ii), which is employed in work in the literature on numerical solution of ordinary differential equations under the name additive compound matrix. Further, the notable related result $C_k(e^{it\mathrm{A}}) = e^{it\Delta_k(\mathrm{A})}$ is proved, and the roles of the $C_k(e^{it\mathrm{A}})$ and $\Delta_k(\mathrm{A})$ in the representation theory of Lie groups and their Lie algebras respectively established.

It is well-known fact that the CE of the parent matrix A can be placed in correspondence with the n-th order difference equation (DE) satisfied by the elements of some sequence X_n of discrete variables $x_r, r = 0, 1, \ldots$. The question arises: what sequences and what DEs are similarly related to the CEs of the $C_k(\mathrm{A})$? The answer is given in terms of certain sets of discrete Hankel variables constructed (in a way described in detail elsewhere) from the x_r.

Keywords: symmetric polynomial, compound matrix, additive compound matrix, characteristic equation, symmetric matrix function, difference equation.

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1 Introduction

The purpose and scope of this article may be explained best with the aid of several definitions in which n is a fixed but arbitrary integer. It sets out from a given matrix A called the parent matrix specified by

Definition 1. A is a symmetric $n \times n$ matrix with eigenvalues

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0, \tag{1}$$

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and characteristic equation CE(A) of degree n

$$\begin{aligned} 0 &= \det(\lambda \mathrm{I}_n - \mathrm{A}) = \prod_{r=1}^{n}(\lambda - \lambda_r) \\ &= \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \ldots + (-1)^n a_n, \end{aligned} \tag{2}$$

where the $a_k = a_k(\lambda_1, \ldots, \lambda_n), \quad k = 1, \ldots, n$, are the elementary symmetric functions (ESFs) [1,2] of the eigenvalues of A, with $\mathrm{tr}\ \mathrm{A} = a_1, \quad \det \mathrm{A} = a_n$.

Definition 2 [3]. Let $Q_{k,n}$ denote the totality of $\left(\begin{array}{c} n \\ k \end{array}\right)$ strictly increasing sequences $\omega = (\omega_1, \ldots, \omega_k)$ of length k of positive integers chosen from $1, \ldots, n$, so that $\omega \in Q_{k,n}$ implies

$$1 \leq \omega_1 < \omega_2 < \ldots < \omega_k \leq n. \tag{3}$$

Definition 3. Let $I = (i_1, \ldots, i_k)$ and $M = (m_1, \ldots, m_k) \in Q_{k,n}$, and let $\xi_M = (\xi_{m_1}, \ldots, \xi_{m_k})$ be a subset of k vectors chosen from a set of n vectors $(\xi_1, \ldots, \xi_n)$ which span $\mathcal{V} = \mathbb{C}^n$. Then the k-vector $\phi(\xi_M)$ may be defined by specifying its components according to

$$\phi_I(\xi_M) = \sum_{\sigma \in \mathcal{S}_k} \epsilon(\sigma) \prod_{j=1}^{k} (\xi_{m_j})_{\sigma(i_j)}, \tag{4}$$

for each $I \in Q_{k,n}$, where the sum is over the permutations σ of parity $\epsilon(\sigma)$ of the symmetric group $\mathcal{S}_k$ on the k elements of I.

Remark 1. There are $\left(\begin{array}{c} n \\ k \end{array}\right)$ linearly independent k-vectors $\phi(\xi_M)$, one for each $M \in Q_{k,n}$, and they span the vector space $\mathcal{V} \wedge \ldots \wedge \mathcal{V} = \bigwedge^{(k)}(\mathcal{V})$.

Compound matrices $C_k(\mathrm{A})$ may now be introduced. The transformation $\mathcal{V} \to \mathcal{V}$ under which

$$\xi_i \mapsto (\mathrm{A}\xi)_i, \quad i = 1, \ldots, n, \tag{5}$$

induces on the k-vector $\phi(\xi_M)$ a transformation $\bigwedge^{(k)}(\mathcal{V}) \to \bigwedge^{(k)}(\mathcal{V})$, given by

$$\phi(\xi_M) \mapsto \phi'(\xi_M) = \phi(\mathrm{A}\xi_{m_1}, \ldots, \mathrm{A}\xi_{m_k}) = C_k(\mathrm{A})\phi(\xi_M), \tag{6}$$

or, in terms of components

$$\phi_I \mapsto \phi'_I = \sum_{J \in Q_{k,n}} C_k(\mathrm{A})_{IJ}\phi_J. \tag{7}$$

This implies

Definition 4. The k-th compound matrix $C_k(\mathrm{A}), \quad k = 0, \ldots, n$, of the symmetric $n \times n$ matrix A has elements given by

$$C_k(\mathrm{A})_{IJ} = \sum_{\sigma \in \mathcal{S}_k} \epsilon(\sigma) \prod_{l=1}^{k} \mathrm{A}_{i_l \sigma(j_l)}, \tag{8}$$

for each pair $\quad I, J \in Q_{k,n}$.

The present article is devoted to the study of the compound matrices $C_k(\mathrm{A}), \quad k = 0, 1, \ldots, n$, [3–5] of the matrix parent A of Definition 1, and of more general symmetric matrix functions of it.

One such family of matrices, the matrices $\mathrm{A}^{(n,k,s)}, \quad s=0,\ldots,k$, is generated by the $C_k(\mathrm{A})$ by means of

Definition 5.

$$C_k(\mathrm{I}+t\mathrm{A}) = \sum_{s=0}^{k} \mathrm{A}^{(n,k,s)} t^s, \quad t \in \mathbb{R}. \tag{9}$$

The special case of Definition 5 in which $s=1$ involves the matrix

$$\mathrm{A}^{(n,k,1)} = \Delta_k(\mathrm{A}) \tag{10}$$

called the k-th additive compound of A [4,6]. It features widely in the literature, see [6] and work cited there, in relation to systems of linear differential equations. Propositions 16 and 17 below offer a different and important view of $\Delta_k(\mathrm{A})$ within the representation theory of a unitary group acting on the vector space $\mathcal{V}$

Useful background material on compound matrices is to be found in [5], and [3, 7, 8] provide a systematic exterior algebra based approach. An extensive listing of properties of the $C_k(\mathrm{A})$, plus some material on the matrices $\mathrm{A}^{(n,k,s)}$ is contained in [4]. In what follows various symmetric functions enter the developments. A clear and detailed elementary account of the theory of them is presented in [1]. It contains many results needed below, including explicit relationships between different types. See also [2], Chapter VI. Appendix A of [9] discusses symmetric functions at a more advanced and general level, as does [10]. Sec. 1.1 summarises well-known definitions of elementary symmetric functions, power sum functions and monomial symmetric functions, and others. The acronyms ESF, PSF and MSF are convenient abbreviations.

The occurrence of compound matrices in graph theory [11] is noted.

To provide a suitable setting for and a viable definition of general symmetric functions of A, begin with reference to a set of $n \times n$ symmetric $\mathrm{A}_l, \quad l=1,\ldots,n$, and introduce a class of matrices that are linear in some subset of them .

Definition 6. Let $L, M \in Q_{k,n}$, and define the matrix

$$\mathrm{M}_k(\mathrm{A}_{l_1},\ldots,\mathrm{A}_{l_k}) \tag{11}$$

by means of its action on $\phi(\xi_M) \in \bigwedge^{(k)}(\mathcal{V})$

$$\mathrm{M}_k(\mathrm{A}_{l_1},\ldots,\mathrm{A}_{l_k})\phi(\xi_{m_1},\ldots,\xi_{m_k}) = \sum_{\sigma\in\mathcal{S}_k} \phi(\mathrm{A}_{\sigma(l_1)}\xi_{m_1},\ldots,\mathrm{A}_{\sigma(l_k)}\xi_{m_k}) \tag{12}$$

noting the absence of signature factors in the sum here.

A matrix of the type (11) is the matrix of an endomorphism of the vector space $\bigwedge^{(k)}(\mathcal{V})$. It is, by construction, totally symmetric under permutations of its k matrix arguments. Attention here is confined to matrices (11) of the form

$$\mathrm{M}_k(\mathrm{A}^{s_1},\ldots,\mathrm{A}^{s_j},\mathrm{I}\ldots), \quad s_1 \ \geq \ldots \geq s_j > 0, \tag{13}$$

in which all the matrix arguments are powers of the parent matrix A. If $j<k$, then the notation implies that the remaining $k-j$ matrix arguments of (13) are each equal to the $n \times n$ unit matrix.

It can be seen to follow from the definitions that

$$\mathrm{A}^{(n,k,s)} = \mathrm{M}_k((\mathrm{A})_s,\mathrm{I},\ldots), \quad s=1,\ldots,k, \tag{14}$$

where the notation $(\mathrm{A})_s$ implies that each of the first s matrix arguments on the right side of (14) is equal to the matrix A. Another special case of (13) is provided by the matrices

$$\mathrm{M}_k(\mathrm{A}^r,\mathrm{I},\ldots), \quad r=1,2,\ldots\ . \tag{15}$$

Some names are clearly needed for the matrices of (13) – (15). They are referred to henceforth in this article, perhaps somewhat loosely, by the (respective) names monomial symmetric matrix function, elementary symmetric matrix function and power sum matrix function of the matrix A. The acronyms MSM, ESM, PSM are convenient. One good reason for the terminology is provided by eigenvalue considerations. Propositions 7 and 11 show that their eigenvalues are given by corresponding MSFs, ESFs and PSFs in the eigenvalues of A. A second one is seen in Proposition 12, which shows that the PSM are exactly the same functions of the ESMs as the corresponding PSF are of the ESFs.

The content of the remainder of the article can now be summarised.

Section 2 introduces the compound matrices $C_k(\mathrm{A})$, reviewing their properties, mainly well-known, and discusses their eigenvalues. Also to achieve element by element equalities of matrices of the sort

$$C_k(\mathrm{A}) = (\det \mathrm{A})C_{n-k}(\mathrm{A}^{-1}) \tag{16}$$

the concept of complementary labelling of elements of a matrices is introduced, since use of it on the right is an essential ingredient of the equality. Section 3 discusses the characteristic equations $\mathrm{CE}(C_k(\mathrm{A}))$ for $k = 1, 2, \ldots$. Section 4 deals with matrices lying beyond the compound matrices in the general scheme of such matrices. Section 4.1 introduces additive compound matrices. If the ξ_i of Definition 3 are functions of a scalar variable t, time say, with evolution in time is governed by

$$\dot{\xi}_i = (\mathrm{A}\xi)_i, \quad i = 1, \ldots, n, \tag{17}$$

then the ordinary differential equation satisfied by the k-vectors of Definition 3 yields the expression (93) for the element $\Delta_k(\mathrm{A})_{IJ}$ for each pair $I, J \in Q_{k,n}$. Section 4.2 contains some comments regarding MSM functions. Section 4.3 contains a detailed discussion of eigenvalues of MSMs. In Sections 5.1 and 5.2 CEs $\mathrm{CE}(A^{(n,k,s)})$ are discussed first in the case in which $k = n - 1$, giving also explicit results in the cases in which

$$(n, k, s) = (4, 2, 1),\ (5, 4, 2),\ (5, 4, 3). \tag{18}$$

Some nice results for determinants of matrices involved are also to be found. In Section 6, wherein A is hermitian, the important result

$$C_k(e^{it\mathrm{A}}) = e^{it\Delta_k(\mathrm{A})}, k = 1, \ldots, n, \quad t \in \mathbb{R}, \tag{19}$$

is proved as Proposition 16. If a Lie group $\mathcal{G}$ acts in the vector space $\mathcal{V}$, and $\mathrm{U} = e^{it\mathrm{A}} \in \mathcal{G}$, then it is shown that the matrices $C_k(\mathrm{U})$ carry a unitary representation of $\mathcal{G}$ for each k, and the matrices $\Delta_k(\mathrm{A})$ yield a hermitian representation of its Lie algebra.

Section 7 develops significant material regarding the relationship of the $\mathrm{CE}(C_k(\mathrm{A}))$ to certain families of difference equations (DEs). The starting point is the correspondence set up in [12] between CE(A) and the $\mathrm{DE}(X_n)$ of Definition 12 below, where X_n is a set of discrete variables $x_r \in X_n$ which obey the $\mathrm{DE}(X_n)$ and suitable initial conditions. The set X_n was introduced in [12] to enable the establishment of explicit expressions for all of the elements of all of the matrices of the QR and QD algorithms. However to achieve the presentation of these expressions in suitably concise form, families of Hankel variables h_k^r, $r = 0, 1, \ldots$, for each $k = 1, \ldots, n$ were introduced, and the question which ultimately led to the present paper was asked. Since the DEs satisfied by the Hankel variables follow from their definition, what are the matrices B_k whose CEs $\mathrm{CE}(\mathrm{B}_k)$ bear the same relation to these DEs as CE(A) bears to $\mathrm{DE}(X_n)$. The answer is $\mathrm{B}_k = C_k(\mathrm{A})$, $k = 1, \ldots, n$, thereby identifying another role played by compound matrices. Section 7.1 gives a brief summary of the role of the Hankel variables in the treatment [12] of the two algorithms.

1.1 Symmetric functions

At various points in this paper, it is necessary to call on results from the theory of symmetric functions, usually of the eigenvalues $\lambda_1, \ldots, \lambda_n$, of a hermitian $n \times n$ matrix A, or of some subset thereof. These include

(i) the elementary symmetric function, ESF, a_k, $\quad k=1,\ldots,n$, defined in (2),
(ii) the power sum, PSF, $S_k=\sum_{j=1}^{n}{\lambda_j}^k$, $\quad k=1,2,\ldots$,
(iii) the monomial symmetric function, MSF,

$$M(i_1\ldots i_n)=\sum\ {\lambda_1}^{i_1}\ldots{\lambda_n}^{i_n}, \tag{20}$$

(iv) the complete symmetric function h_k, which is the sum of all the monomial symmetric functions of degree k,
(v) the Schur function, which can be viewed as a $GL(n)$ character.

The usual definition of the MSF of (iii) of degree $I=\sum_{j=1}^{n}i_j$, when $n\geq i_1\geq i_2\geq\ldots\geq i_n\geq 0$, is as follows. Set out from the single term ${\lambda_1}^{i_1}\ldots{\lambda_n}^{i_n}$. If this is acted on by all permutations of the symmetric group $\mathcal{S}_n$ on the λ's, then $M(i_1\ldots i_n)$ is the sum of all the distinct terms so generated, each counted only once.

It is sometimes useful in order to make patterns clear to write the following in MSF notation

$$a_1=M(1),\quad a_2=M(11),\quad \text{etc., and}\quad S_k=\sum\lambda^k=M(k). \tag{21}$$

Actually the MSFs that enter the work below, especially in Section 3.5, do so in a somewhat different way from the above, one whose equivalence to it can readily be verified: see the appendix for detail.

To define a Schur function, first let $\mu=\mu_n\mu_{n-1}\ldots\mu_2\mu_1$ denote a Young tableau for which the rows have lengths

$$\mu_n\geq\mu_{n-1}\ldots\geq\mu_1\geq 0. \tag{22}$$

Then [13] the corresponding $GL(n)$ character is given by

$$\chi_\mu(\lambda_1,\ldots,\lambda_n)=\frac{\det\, v_\mu(\lambda_1,\ldots,\lambda_n)}{\det\, v_0(\lambda_1,\ldots,\lambda_n)}. \tag{23}$$

Here the quantities in the numerator and denominator are the determinants of the $n\times n$ matrices v_μ and v_0 with elements

$$(v_\mu)_{ij}={\lambda_i}^{\mu_j+j-1},\quad 1\leq i,j\leq n, \tag{24}$$

and

$$(v_0)_{ij}={\lambda_i}^{j-1},\quad 1\leq i,j\leq n. \tag{25}$$

The latter matrix has as its determinant the Vandermonde determinant

$$v_0(\lambda_1,\ldots,\lambda_n)=\prod_{1\leq i<j\leq n}(\lambda_j-\lambda_i), \tag{26}$$

The character $\chi_\mu(\lambda_1,\ldots,\lambda_n)$ is referred to below as a Schur function, using the notation $\{\mu_n,\ldots,\mu_1\}$.

The policy adopted in this paper is to express all functions invariant under $\mathcal{S}$ in terms of the ESFs a_k. Theorem 13.12 of [14] supports this. Its statement is: every polynomial symmetric function of the variables $\lambda_1,\lambda_2,\lambda_3,\ldots$, is actually a (unique [2])) polynomial in the $a_1,a_2,a_3,\ldots$.

An appendix collects a few well-known results expressing MSFs in terms of ESFs, and indicates that easy programming yields the further results of this sort needed in Sections 5.1 and 5.2.

2 Compound matrices

The elements of the k-th compound matrix $C_k(\mathrm{A}), k=0,\ldots,n$, are given by (8) in Definition 4.

Example 1. There are $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ bivectors $\phi(\xi,\eta)$ one for each pair of vectors ξ,η belonging to any given set of n vectors spanning $\mathcal{V}$. These have components

$$\phi_{i\alpha}=\xi_i\eta_\alpha-\xi_\alpha\eta_i,\quad 1\leq i<\alpha\leq n. \tag{27}$$

The transformation law

$$\phi(\xi,\eta) \mapsto \phi'(\xi,\eta) = \phi(\mathrm{A}\xi, \mathrm{A}\eta) = C_k(\mathrm{A})\phi(\xi,\eta) \tag{28}$$

induced by (5), reads in terms of components as

$$\phi_{i\alpha} \mapsto {\phi_{i\alpha}}' = \sum_{1\le j<\beta\le n} C_2(\mathrm{A})_{i\alpha,j\beta}\phi_{j\beta}, \quad i<\alpha, \tag{29}$$

This yields the definition

$$\begin{aligned} C_2(\mathrm{A})_{i\alpha,j\beta} &= (\mathrm{A}_{ij}\mathrm{A}_{\alpha\beta} - \mathrm{A}_{\alpha j}\mathrm{A}_{i\beta}), \quad i<\alpha, j<\beta, \\ &= \begin{vmatrix} \mathrm{A}_{ij} & \mathrm{A}_{i\beta} \\ \mathrm{A}_{\alpha j} & \mathrm{A}_{\alpha\beta} \end{vmatrix}, \quad i<\alpha, j<\beta. \end{aligned} \tag{30}$$

Note also that the unit matrix $C_2(\mathrm{I})$ has components

$$C_2(\mathrm{I})_{i\alpha,j\beta} = (\delta_{ij}\delta_{\alpha\beta} - \delta_{\alpha j}\delta_{i\beta}), \tag{31}$$

and satisfies

$$C_2(\mathrm{I})C_2(\mathrm{A}) = C_2(\mathrm{A})C_2(\mathrm{I}) = C_2(\mathrm{A}). \tag{32}$$

Similarly

$$\begin{aligned} C_3(\mathrm{A})_{i\alpha a,j\beta b} &= \mathrm{A}_{ij}\mathrm{A}_{\alpha\beta}\mathrm{A}_{ab} + \text{signed permutations of} \quad (j\beta b) \\ &= \begin{vmatrix} \mathrm{A}_{ij} & \mathrm{A}_{i\beta} & \mathrm{A}_{ib} \\ \mathrm{A}_{\alpha j} & \mathrm{A}_{\alpha\beta} & \mathrm{A}_{\alpha b} \\ \mathrm{A}_{aj} & \mathrm{A}_{a\beta} & \mathrm{A}_{ab} \end{vmatrix}, \end{aligned} \tag{33}$$

for $i<\alpha<a, \quad j<\beta<b$.

Remark 2. For any n, the cases $r = 0, 1, n-1, n$ may be noted

$$C_0(\mathrm{A}) = 1, \quad C_1(\mathrm{A}) = \mathrm{A}, \quad C_{n-1}(\mathrm{A}) = \text{adj A} = (\det \mathrm{A})\ \mathrm{A}^{-1}, \quad C_n(\mathrm{A}) = (\det \mathrm{A})\ 1. \tag{34}$$

This is the full set of compound matrices for $n = 3$, while for $n = 4$ the first instance arises of a compound matrix not familiar from elementary work: $C_2(\mathrm{A})$ of dimension 6.

The statement

$$C_{n-1}(\mathrm{A}) = (\det \mathrm{A})C_1(\mathrm{A}^{-1}) \tag{35}$$

implied by (34) clearly needs careful attention for the labellings of elements on the two sides are not as things stand compatible. This gives rise to the content of Section 2.2 .

The eigenvectors $e^{(r)}$ of A belonging to the eigenvalues $\lambda_r, \quad r = 1,\ldots,n$, satisfy

$$\sum_{j=1}^{n} \mathrm{A}_{ij}e_j^{(r)} = \lambda_r e_i^{(r)}, \quad i = 1,\ldots,n. \tag{36}$$

It is then easy to establish [3, 15]

Proposition 1. For each $R = (r_1,\ldots,r_k) \in Q_{k,n}$, the eigenvalue λ_R belonging to the eigenvector $\phi(e^R)$ of the matrix $C_k(\mathrm{A})$ is given by

$$\sum_{J\in Q_{k,n}} C_k(\mathrm{A})_{IJ}\phi_J(e^R) = \lambda_R\phi_I(e^R), \quad I \in Q_{k,n}, \tag{37}$$

where

$$\lambda_R = \lambda_{r_1}\ldots\lambda_{r_k}, \tag{38}$$

and

$$\phi_I(e^R) = \phi_I(e^{(r_1)}, \ldots, e^{(r_k)}). \tag{39}$$

Since $R \in Q_{k,n}$, implies

$$1 \leq r_1 < r_2 < \ldots < r_k \leq n, \tag{40}$$

the number of distinct eigenvalues, $\left(\begin{array}{c} n \\ k \end{array} \right)$, is equal to dim $C_k(\mathrm{A})$.

2.1 Properties of compound matrices

Proposition 2. This lists some well-known results from [3–5] involving the compound matrices $C_k(\mathrm{A}), \quad k = 1, \ldots, n$, of a symmetric $n \times n$ matrix A

$$\begin{aligned}
\dim C_k(\mathrm{A}) &= \left(\begin{array}{c} n \\ k \end{array} \right) \\
C_k(t\mathrm{A}) &= t^k C_k(\mathrm{A}), \quad t \in \mathbb{R} \\
C_k(\mathrm{A})^T &= C_k(\mathrm{A}) && (41) \\
\det C_k(\mathrm{A}) &= (\det \mathrm{A})^{N(k)}, \quad N(k) = \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right) && (42) \\
\mathrm{tr}\ C_k(\mathrm{A}) &= a_k && (43) \\
[C_k(\mathrm{A})]^r &= C_k(\mathrm{A}^r), \quad r = 1, 2, \ldots. && (44)
\end{aligned}$$

Eq. (42) is the Sylvester-Francke theorem [4,5], Eq. (43) follows from Proposition 1, while Eq. (44) follows from the Binet-Cauchy theorem [4,5]

$$C_k(\mathrm{A})C_k(\mathrm{B}) = C_k(\mathrm{AB}), \tag{45}$$

where B is another $n \times n$ matrix.

It is easy to check (43) for $k = 2$:

$$\begin{aligned}
\mathrm{tr}\ C_2(\mathrm{A}) &= \sum_{i<\alpha} (\mathrm{A}_{ii}\mathrm{A}_{\alpha\alpha} - \mathrm{A}_{i\alpha}\mathrm{A}_{\alpha i}) \\
&= \frac{1}{2}\left[(\mathrm{trA})^2 - \mathrm{trA}^2\right] = \frac{1}{2}({S_1}^2 - S_2) = a_2
\end{aligned}$$

Further development of the trace result (43) will be of major technical assistance in obtaining explicit forms of the CE($C_k(\mathrm{A})$). What is needed are the well-known formulas for the ESFs a_r in terms of power sum functions. These include

$$\begin{aligned}
\mathrm{tr}\ C_2(\mathrm{A}) &= \frac{1}{2}({S_1}^2 - S_2) && (46) \\
\mathrm{tr}\ C_3(\mathrm{A}) &= \frac{1}{6}\left({S_1}^3 - 3S_1S_2 + 2S_3\right) && (47) \\
\mathrm{tr}\ C_4(\mathrm{A}) &= \frac{1}{24}({S_1}^4 - 6{S_1}^2S_2 + 8S_1S_3 + 3{S_2}^2 - 6S_4) && (48)
\end{aligned}$$

where

$$S_k = \mathrm{tr}\ \mathrm{A}^k = \sum_{i=1}^{n} {\lambda_i}^k, \tag{49}$$

and have the consequences

$$\begin{aligned}
\mathrm{tr}\ C_2(\mathrm{A}^j) &= \frac{1}{2}(S_j^2 - S_{2j}) && (50) \\
\mathrm{tr}\ C_3(\mathrm{A}^j) &= \frac{1}{6}\left({S_j}^3 - 3S_jS_{2j} + 2S_{3j}\right), && (51)
\end{aligned}$$

and so on, for $j = 2, 3, \ldots$.

2.2 Complementary labelling of matrix elements

Proposition 3. If A is a symmetric $n \times n$ matrix then

$$C_k(\mathrm{A})_{JI} = (\det \mathrm{A}) C_{n-k}(\mathrm{A}^{-1})_{\bar{I}\bar{J}} \sigma(I\bar{I})\sigma(J\bar{J}) \tag{52}$$

for each $I, J \in Q_{k,n}$. Here

$$\bar{I} = (\bar{i}_1, \ldots, \bar{i}_{n-k}) \quad 1 \leq \bar{i}_1 < \ldots < \bar{i}_{n-k} \leq n \tag{53}$$

denotes the complement $\bar{I} \in Q_{k-n,n}$ of $I = (i_1, \ldots, i_k) \in Q_{k,n}$ in $(1, \ldots, n)$, and $\sigma(I\bar{I})$ is the signature of the permutation in $(I\bar{I})$ of $(1, \ldots, n)$. Similarly for J and $\bar{J}$.

To develop an understanding of the situation, consider

Definition 7. The complemantary labelling of the elements of $C_k(\mathrm{A})$ is related to the standard labelling $C_k(\mathrm{A})_{IJ}$ by

$$C_k(\mathrm{A})_{\bar{J}\bar{I}} = C_k(\mathrm{A})_{IJ} \sigma(I\bar{I})\sigma(J\bar{J} \tag{54}$$

in the language of Proposition 3.

Proposition 4. The identity

$$(\det \mathrm{A}) C_{n-k}(\mathrm{A}^{-1}) = C_k(\mathrm{A}), \tag{55}$$

is valid, provided that complementary labelling is used on the left side.

A general proof is not included but the treatment of a single example should suffice to make clear where the above ideas come from. Eq. (35) is the $k = n-1$ case of (55).

Example 2. For 7×7 symmetric A, compare $C_4(\mathrm{A})$ and $\Delta\ C_3(\mathrm{A}^{-1})$ where $\Delta = \det \mathrm{A}$. The matrices have the same spectra and are therefore equivalent. This follows

$$\lambda_R = \frac{\Delta}{\lambda_{\bar{R}}}, \tag{56}$$

where $\bar{R} \in Q_{3,7}$ is the complement in $(1, \ldots, 7)$ of $R \in Q_{4,7}$. The labelling of the elements of the two matrices so that they are equal element by element is illustrated by treatment of one typical case.

Writing $\mathrm{A} = [a_{i\,j}]$ and $\mathrm{C} = [c_{i\,j}]$ where c_{ij} is the cofactor of a_{ji} in A, it is seen that

$$\Delta(\mathrm{A}^{-1})_{ij} = \mathrm{C}_{ij}, \quad 1 \leq i, j \leq 7. \tag{57}$$

Temporarily and only in this paragraph, let $c_\alpha, a_\beta, e_\gamma$ denote the α-th, β-th, γ-th columns of $\mathrm{C}, \mathrm{A}, \mathrm{I}_7$, and take the determinant of the obvious matrix identity

$$\mathrm{A}\,[c_2 c_5 c_7 e_1 e_4 e_5 e_7] = [(\Delta e_2)(\Delta e_5)(\Delta e_7) a_1 a_4 a_5 a_7]\,. \tag{58}$$

This gives rise to

$$\Delta \begin{vmatrix} c_{22} & c_{25} & c_{27} \\ c_{32} & c_{35} & c_{37} \\ c_{62} & c_{65} & c_{67} \end{vmatrix} \sigma(2361457) = \Delta^3 \begin{vmatrix} a_{11} & a_{14} & a_{15} & a_{17} \\ a_{31} & a_{34} & a_{35} & a_{37} \\ a_{41} & a_{44} & a_{45} & a_{47} \\ a_{61} & a_{64} & a_{65} & a_{67} \end{vmatrix} \sigma(2571346). \tag{59}$$

The two entries $\sigma(...)$ here are the signature factors, one plus and one minus in this example, that arise naturally from expanding the 7×7 determinants arising from (58). It follows now that

$$C_4(\mathrm{A})_{1346,1457} = (-1)\Delta C_3(\mathrm{A}^{-1})_{236,257}. \tag{60}$$

Noting that 236 is the complement of 1457 and that 257 is that 1346, (60) is seen to agree exactly with the requirements of Proposition 4.

For the case of 3×3 A, the elements $C_2(\mathrm{A})_{i\alpha,j\beta}$ given by (30) are all seen to coincide to within a sign with cofactors in A. Introduce the complementary labelling by

$$C_2(\mathrm{A})_{\bar{b}\bar{a}} = \sigma(i\alpha a)\sigma(j\beta b)C_2(\mathrm{A})_{i\alpha,j\beta}, \tag{61}$$

where $i\alpha = (12, 13, 23) \in Q_{2,3}$ have complements in $(1, 2, 3)$ which are equal to $(\overline{12} = \bar{3}, \overline{13} = \bar{2}, \overline{23} = \bar{1}) \in Q_{1,3}$. Labelling the rows and columns of $C_2(\mathrm{A})_{\bar{b}\bar{a}}$ in their natural $Q_{1,3}$ order finally admits presenting the element by element equality of matrices

$$C_2(\mathrm{A})_{\bar{i}\bar{j}} = \mathrm{C}_{ij} \tag{62}$$

or, since each $\bar{k}$ denotes the same (numerical) value $1, 2, 3$ as does the corresponding k,

$$C_2(\mathrm{A}) = \mathrm{C} = \Delta\mathrm{A}^{-1}, \tag{63}$$

3 Characteristic equations

For the symmetric $n \times n$ matrix A, the characteristic equation CE(A) of Definition 1 can be written in further useful ways, including

$$\begin{aligned} 0 &= \sum_{k=0}^{n} (-)^k \lambda^{n-k} a_k, \quad a_0 = 1, \\ &= \sum_{k=0}^{n} (-)^k \lambda^{n-k} \mathrm{tr}\, C_k(\mathrm{A}). \end{aligned} \tag{64}$$

The second form [1, 3, 4, 8] can be seen from (43) to agree with the first. A useful third form can be developed, which expresses the ESFs a_k in terms of the PSFs S_l.

Definition 8. Let $\rho(P_r)$ denote the class of elements of the symmetric group $\mathcal{S}_r$ on r symbols, with cycle structure corresponding to the partition P_r of r

$$P_r = \{1^{r_1}\, 2^{r_2}\, 3^{r_3} \ldots r^{r_r}\}, \tag{65}$$

where

$$r = r_1 + 2r_2 + 3r_3 + \ldots + r\, r_r. \tag{66}$$

Then let

$$S(P_r) = {S_1}^{r_1} {S_2}^{r_2} \ldots {S_r}^{r_r}, \tag{67}$$

so that, see *e.g.* [2, 16],

$$r! a_r = \sum_{P_r} \sigma(P_r) h(P_r) S(P_r) \tag{68}$$

where $\sigma(P_r)$ and $h(P_r)$ are the parity and order of $\rho(P_r)$, there being one term in the sum for each class or partition. Use of well-known results for $\sigma(P_r)$ and $h(P_r)$ leads to the more explicit result

Proposition 5.

$$a_r = (-1)^r \sum_{P_r} \prod_{i=1}^{r} \Phi_i, \quad \Phi_i = \frac{1}{r_i!}(-\frac{1}{i} S_i)^{r_i}. \tag{69}$$

A systematic approach to obtaining characteristic equations CE(C_k(A)) can be described. Use (69) to write down the first few terms of CE(A) for some $N \times N$ matrix A explicitly :

$$\begin{aligned} 0 &= \lambda^N - \frac{1}{2}(S_1^2 - S_2)\lambda^{N-1} + \frac{1}{6}({S_1}^3 - 3S_1S_2 + 2S_3)\lambda_{N-2} \\ &\quad -\frac{1}{24}({S_1}^4 - 6{S_1}^2S_2 + 8S_1S_3 + 3{S_2}^2 - 6S_4)\lambda^{N-3} + \ldots . \end{aligned} \tag{70}$$

To write down the CE(C_k(A)) when A is $n \times n$ and CE(C_k(A)) is $N \times N$ where $N = \binom{n}{k}$, it is necessary only to apply to (70), the replacement of $S_j = \text{tr } \mathrm{A}^j$, throughout by

$$\Sigma_j = \text{tr } C_k(\mathrm{A}^j), \tag{71}$$

which is the sum over the j-th powers of the eigenvalues of C_k(A). This furnishes a general, if not genuinely tractable, way of obtaining the CE(C_k(A)).

3.1 Case of C_2(A)

Example 3. Consider C_2(A) for $n = 4$. This is a 6×6 matrix with six distinct eigenvalues $\mu_r, \quad r = 1, 2, \ldots, 6$, explicitly

$$\lambda_1\lambda_2, \quad \lambda_1\lambda_3, \quad \lambda_1\lambda_4, \quad \lambda_2\lambda_3, \quad \lambda_2\lambda_4, \quad \lambda_3\lambda_4. \tag{72}$$

Writing CE(C_2(A)) in the form

$$0 = \lambda^6 - f_1\lambda^5 + f_2\lambda^4 - f_3\lambda^3 + f_4\lambda^2 - f_5\lambda + f_6, \tag{73}$$

there are some easy results

$$\begin{aligned} f_1 &= \text{tr } C_1(C_2(\mathrm{A})) = \text{tr } C_2(\mathrm{A}) = a_2, \\ f_6 &= \det C_2(\mathrm{A}) = (\det \mathrm{A})^3 = (a_4)^3 \\ f_5 &= \text{tr adj } (C_2(\mathrm{A}) = (a_4)^3(a_2/a_4) = a_2{a_4}^2. \end{aligned} \tag{74}$$

To proceed further, consider the replacement S_k by $\Sigma_k = \sum_{j=1}^{6} {\mu_j}^k$ throughout the CE of some 6×6 symmetric A. This gives rise to

$$\begin{aligned} f_2 &= \frac{1}{2}({\Sigma_1}^2 - \Sigma_2) = \frac{1}{2}\left[(\text{tr } C_2(\mathrm{A}))^2 - \text{tr } C_2(\mathrm{A}^2)\right] \\ &= \frac{1}{8}({S_1}^2 - S_2)^2 - \frac{1}{4}({S_2}^2 - S_4) = a_1a_3 - a_4, \end{aligned} \tag{75}$$

and f_3 and f_4 are evaluated in a similar way. Hence CE(C_2(A)) for $n = 4$ is

$$\begin{aligned} 0 &= \lambda^6 - a_2\lambda^5 + (a_1a_3 - a_4)\lambda^4 - ({a_3}^2 - 2a_2a_4 + a_4{a_1}^2)\lambda^3 \\ &\quad +(a_1a_3 - a_4)a_4\lambda^2 - a_2{a_4}^2\lambda + {a_4}^3. \end{aligned} \tag{76}$$

Example 4. More generally, for C_2(A) for $n \times n$ A, so that $a_r, \quad r = 1, 2, \ldots, n$, enter, a characteristic equation of degree $N = \begin{pmatrix} n \\ 2 \end{pmatrix}$ is found

$$\begin{aligned} 0 &= \lambda^N - a_2\lambda^{N-1} + (a_1a_3 - a_4)\lambda^{N-2} \\ &\quad - \left[({a_3}^2 - 2a_2a_4 + a_4{a_1}^2) - a_1a_5 + a_6\right]\lambda^{N-3} \\ &\quad + \left[(a_1a_3 - a_4)a_4 + a_5({a_1}^3 - 3a_1a_2 + a_3)\right. \\ &\quad \left. -a_6({a_1}^2 - 2a_2) + a_1a_7 - a_8\right]\lambda^{N-4} \\ &\quad + \ldots - (-1)^N a_{n-2}{a_n}^{n-2}\lambda + (-1)^N {a_n}^{n-1}. \end{aligned} \tag{77}$$

Note here that the pairs (a_3, a_4), (a_5, a_6), (a_7, a_8) enter first for the third, fourth, fifith terms and so on. If $n = 4$ so that a_r, $r > 4$, are absent, (77) reduces as required to give agreement with (76). If $n = 3$, so that only a_1, a_2, a_3 enter, then CE(C_2(A)) given by

$$0 = \lambda^3 - a_2\lambda^2 + a_1a_3\lambda - {a_3}^2 \tag{78}$$

agrees with CE(adj A).

3.2 Case of C_3(A)

Example 5. The characteristic equation of C_3(A) for $n \times n$ A is of degree $\begin{pmatrix} n \\ 3 \end{pmatrix}$. To obtain it from the CE of some matrix A of dimension $N = \begin{pmatrix} n \\ 3 \end{pmatrix}$, the replacement of

$$S_k \quad \text{by} \quad \Sigma_k = \text{tr } C_3(\text{A})^k = \text{tr } C_3(\text{A}^k) \tag{79}$$

is used. The early terms of some output from computer algebra are

$$\begin{aligned} 0 \quad = \quad & \lambda^N - a_3\lambda^{N-1} + (a_2a_4 - a_1a_5 + a_6)\lambda^{N-2} \\ & - \big[a_1{a_4}^2 + ({a_2}^2 - a_4 - 2a_1a_3)a_5 \\ & \quad +(3a_3 - a_1a_2)a_6 + ({a_1}^2 - a_2)a_7 - a_1a_8 + a_9\big]\, \lambda^{N-3} \\ & \quad + \big[{a_4}^3 + \ldots\big]\, \lambda^{N-4} - \ldots . \end{aligned} \tag{80}$$

Note that the triples (a_4, a_5, a_6), (a_7, a_8, a_9) enter first for the third, fourth terms and so on. For $n = 4$, when a_r, $r > 4$, are absent, and (80) reduces to

$$0 = \lambda^4 - a_3\lambda^3 + a_2a_4\lambda^2 - a_1{a_4}^2\lambda + {a_4}^3, \tag{81}$$

as is expected for $C_3(\text{A}) = \text{adj A}$, and easily checked directly.

3.3 Some further examples

If A is 5×5, then C_2(A) and C_3(A) are of dimension $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$. The CE($C_2$(A)) is

$$\begin{aligned} 0 \quad = \quad & \lambda^{10} - a_2\lambda^9 + (a_1a_3 - a_4)\lambda^8 - \big[({a_3}^2 - 2a_2a_4 + {a_1}^2a_4) - a_1a_5\big]\, \lambda^7 \\ & + \big[(a_1a_3 - a_4)a_4 + (a_3 - 3a_1a_2 + {a_1}^3)a_5\big]\, \lambda^6 \\ & - \big[a_2{a_4}^2 + ({a_1}^2a_3 - 2a_1a_4 - 2a_2a_3)a_5 + 2{a_5}^2\big]\, \lambda^5 \\ & + \big[{a_4}^3 + (a_2 - {a_1}^2){a_5}^2 + (a_1a_2 - 3a_3)a_4a_5\big]\, \lambda^4 \\ & - \big[a_1{a_4}^2 + ({a_2}^2 - 2a_1a_3 - a_4)a_5\big]\, a_5\lambda_3 \\ & +(a_2a_4 - a_1a_5){a_5}^2\lambda^2 - a_3{a_5}^3\lambda + {a_5}^4. \end{aligned} \tag{82}$$

If this is written

$$0 = \sum_{r=0}^{10} (-1)^r f_r \lambda^{10-r}, \quad \text{with} \quad f_0 = 1, \quad f_r = \text{tr } C_r(C_2(\text{A})), \tag{83}$$

then CE(C_3(A)) is given by

$$0 = \sum_{r=0}^{10} (-1)^r g_r \lambda^{10-r}, \quad \text{with} \quad g_0 = 1, \quad g_r = \text{tr } C_r(C_3(\text{A})), \tag{84}$$

where

$$g_r = f_{10-r}{a_5}^{r-4}, \quad r = 0, \dots, 10. \tag{85}$$

Remark 3. There is a related result for $n = 4$ and $C_2(\mathrm{A})$ of dimension six. If $\mathrm{CE}(C_2(\mathrm{A}))$ is written as

$$0 = \sum_{r=0}^{6} (-1)^r f_r \lambda^{6-r}, \quad \text{with} \quad f_0 = 1, \tag{86}$$

then the result (76) conforms to

$$f_r = f_{6-r}{a_4}^{r-3}, \quad r = 0, 1, \dots, 6. \tag{87}$$

Results like those of (85) and (87) may alos be inferred from consideration of the eigenvalues of the compound matrices involved.

4 ES, MS and PS matrix functions

4.1 Additive compound matrices

For a given symmetric matrix A, the k-th additive compound $\Delta_k(\mathrm{A})$, sometimes referred to simply as a compound matrix, features widely in the literature, often in relation to systems of differential equations: see [6] and references therein. The distinction between $\Delta_k(\mathrm{A})$ and $C_k(\mathrm{A})$ is explicit and clear in [4], as well as in the following important special case of Definition 5.

Definition 9. The k-th additive compound $\Delta_k(\mathrm{A})$ of the symmetric $n \times n$ matrix A is defined by

$$\Delta_k(\mathrm{A}) = \mathrm{A}^{(n,k,1)} = \frac{d}{dt} C_k(\mathrm{I} + t\mathrm{A})\bigg|_{t=0}. \tag{88}$$

Such matrices arise when the vector arguments of the k-vector $\phi(\xi_1, \dots, \xi_k)$ depend on time t, according to

$$\dot{\xi}_i = (\mathrm{A}\xi)_i, \quad i = 1, \dots, n. \tag{89}$$

Proposition 6. The k-vector $\phi(\xi_1, \dots, \xi_k)$ obeys the differential equation

$$\begin{aligned} \frac{d\phi}{dt} &= \Delta_k(\mathrm{A})\phi(\xi_1, \dots, \xi_k) \\ &= \sum_{j=1}^{k} \phi(\xi_1, \dots, \xi_{j-1}, \mathrm{A}\xi_j, \xi_{j+1}, \dots, \xi_k). \end{aligned} \tag{90}$$

Proof. This follows immediately from the linearity of $\phi(\xi_1, \dots, \xi_k)$ in its arguments. □

It follows (90) that

$$\Delta_k(\mathrm{A}) = \mathrm{M}_k(\mathrm{A}, \mathrm{I}, \dots, \mathrm{I}), \tag{91}$$

Also, writing

$$\frac{d\phi_I}{dt} = \sum_{J \in Q_{k,n}} \Delta_k(\mathrm{A})_{IJ}\phi_J, \quad I \in Q_{k,n}, \tag{92}$$

it can be seen that, for each $I, J \in Q_{k,n}$

$$\begin{aligned} \Delta_k(\mathrm{A})_{IJ} \quad = \quad & [\mathrm{A}_{i_1j_1}\delta_{i_2j_2} \dots \delta_{i_kj_k} + \delta_{i_1j_1}\mathrm{A}_{i_2j_2} \dots \delta_{i_kj_k} + \dots + \delta_{i_1j_1}\delta_{i_2j_2} \dots \mathrm{A}_{i_kj_k}] \\ & + \text{signed permutations of} \quad (j_1, \dots, j_k). \end{aligned} \tag{93}$$

Example 6. The $k=2$ case

$$\Delta_2(\mathrm{A})_{i\alpha,\, j\beta} = \delta_{ij}\mathrm{A}_{\alpha\beta} + \mathrm{A}_{ij}\delta_{\alpha\beta} - \delta_{\alpha j}\mathrm{A}_{i\beta} - \mathrm{A}_{\alpha j}\delta_{i\beta}, \quad i<\alpha,\ \ j<\beta, \tag{94}$$

agrees for $n=4$ with the 6×6 matrix displayed in Eq. (2.8) of [6].

To gain further information about the algebra of matrices of the type $\Delta_k(\mathrm{A})$ – for use, for example, in relation to higher time-derivatives of the k-vectors $\phi(\xi_1,\ldots,\xi_k)$ – requires that $\Delta_k(\mathrm{A})$ be placed in the more general setting [4] of Definition 5. In this context note

$$C_k(\mathrm{I}) = \mathrm{A}^{(n,k,0)} \tag{95}$$

is the unit matrix in $\bigwedge^{(k)}(\mathcal{V})$,

$$\begin{aligned} \Delta_k(\mathrm{A}) &= \mathrm{A}^{(n,k,1)} \\ C_k(\mathrm{A}) &= \mathrm{A}^{(n,k,k)}. \end{aligned} \tag{96}$$

The first of these is consistent with (91) because of (14). Also, if $k=n$,

$$C_n(\mathrm{I}+t\mathrm{A}) = \det\,(\mathrm{I}+t\mathrm{A}), \tag{97}$$

so that $\mathrm{A}^{(n,n,s)} = a_s, \quad s=1,\ldots,n$.

4.2 Monomial symmetric matrix functions

Recall Definition 6 of the monomial symmetric matrix function of powers of the parent matrix A, and certain special cases noted above. This section then collects a few easy results involving MSMs.

Example 7.

$$\begin{aligned} \mathrm{A}^{(n,3,0)} &= C_3(\mathrm{I}_n) = \mathrm{I}_N, \quad N = \binom{n}{3} \\ \mathrm{A}^{(n,3,1)} &= \Delta_3(\mathrm{A}) = \mathrm{M}_3(\mathrm{A},\mathrm{I},\mathrm{I}) \\ \mathrm{A}^{(n,3,2)} &= \mathrm{M}_3(\mathrm{A},\mathrm{A},\mathrm{I}) \ = \ \mathrm{M}_3((\mathrm{A})_2,\mathrm{I}) \\ \mathrm{A}^{(n,3,3)} &= \mathrm{M}_3(\mathrm{A},\mathrm{A},\mathrm{A}) \ = \ \mathrm{M}_3((\mathrm{A})_3) = C_3(\mathrm{A}) \end{aligned} \tag{98}$$

In terms of components

$$\begin{aligned} \mathrm{A}^{(n,3,1)}{}_{IJ} &= \sum_{\sigma\in\mathcal{S}_3} \epsilon(\sigma)\left(\mathrm{A}_{i_1\sigma(j_1)}\delta_{i_2\sigma(j_2)}\delta_{i_3\sigma(j_3)} + \delta_{i_1\sigma(j_1)}\mathrm{A}_{i_2\sigma(j_2)}\delta_{i_3\sigma(j_3)} + \delta_{i_1\sigma(j_1)}\delta_{i_2\sigma(j_2)}\mathrm{A}_{i_3\sigma(j_3)}\right) \\ \mathrm{A}^{(n,3,2)}{}_{IJ} &= \sum_{\sigma\in\mathcal{S}_3} \epsilon(\sigma)\left(\delta_{i_1\sigma(j_1)}\mathrm{A}_{i_2\sigma(j_2)}\mathrm{A}_{i_3\sigma(j_3)} + \mathrm{A}_{i_1\sigma(j_1)}\delta_{i_2\sigma(j_2)}\mathrm{A}_{i_3\sigma(j_3)} + \mathrm{A}_{i_1\sigma(j_1)}\mathrm{A}_{i_2\sigma(j_2)}\delta_{i_3\sigma(j_3)}\right). \end{aligned} \tag{99}$$

Consideration of actions in $\bigwedge^{(k)}(\mathcal{V})$ provides an easy way to obtain product laws for MSM functions of A, and hence, in virtue of Proposition 11 below, also of product laws of MSFs.

Example 8. Recall the identification $\mathrm{A}^{(n,k,s)} = \mathrm{M}_k(\mathrm{A},\ldots,\mathrm{A},\mathrm{I},\ldots))$ with $s, \quad s=1,\ldots,k$ arguments A, and $k-s$ arguments I, the special case $\Delta_k(\mathrm{A}) = \mathrm{A}^{(n,k,1)}$, and $\Delta_k(\mathrm{A}^r) = \mathrm{M}_k(\mathrm{A}^r,\mathrm{I},\ldots)$.

First prove the identity

$$\mathrm{M}_k(\mathrm{A},\mathrm{I},\ldots)^2 = \mathrm{M}_k(\mathrm{A}^2,\mathrm{I},\ldots) + 2\mathrm{M}_k(\mathrm{A},\mathrm{A},\mathrm{I},\ldots) \tag{100}$$

which leads to

$$\left(\begin{array}{c} \mathrm{M}_k(\mathrm{A},\mathrm{A}) \\ \mathrm{M}_k(\mathrm{A}^2,\mathrm{I}) \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ 1 & -2 \end{array}\right) \left(\begin{array}{c} \mathrm{A}^{(n,k,1)2} \\ \mathrm{A}^{(n,k,2)} \end{array}\right). \tag{101}$$

An independent proof of (100) in the form

$$\Delta_k(\mathrm{A}^2) = \Delta_k(\mathrm{A})^2 - 2\mathrm{A}^{(n,k,2)} \tag{102}$$

can be obtained by differentiation of the two sides of (92) with respect to t.

Next prove the identity

$$\mathrm{M}_k(\mathrm{A},\mathrm{I},\ldots)\mathrm{M}_k(\mathrm{A},\mathrm{A},\mathrm{I},\ldots) = \mathrm{M}_k(\mathrm{A}^2,\mathrm{A},\mathrm{I},\ldots) + 3\mathrm{M}_k(\mathrm{A},\mathrm{A},\mathrm{A},\mathrm{I},\ldots). \tag{103}$$

This provides the second line of

$$\left(\begin{array}{c} \mathrm{M}_k(\mathrm{A},\mathrm{A},\mathrm{A}) \\ \mathrm{M}_k(\mathrm{A}^2,\mathrm{A},\mathrm{I}) \\ \mathrm{M}_k(\mathrm{A}^3,\mathrm{I},\mathrm{I}) \end{array}\right) = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 3 \end{array}\right) \left(\begin{array}{c} \mathrm{A}^{(n,k,1)3} \\ \mathrm{A}^{(n,k,1)}\mathrm{A}^{(n,k,2)} \\ \mathrm{A}^{(n,k,3)} \end{array}\right). \tag{104}$$

To get the third line, use

$$\mathrm{M}_k(\mathrm{A}^2,\mathrm{I},\ldots)\mathrm{M}_k(\mathrm{A},\mathrm{I},\ldots) = \mathrm{M}_k(\mathrm{A}^3,\mathrm{I},\ldots) + \mathrm{M}_k(\mathrm{A}^2,\mathrm{A},\mathrm{I},\ldots) \tag{105}$$

and substitute for $\mathrm{M}_k(\mathrm{A}^2,\mathrm{I})$ and $\mathrm{M}_k(\mathrm{A}^2,\mathrm{A},\mathrm{I})$ using (103) and (105).

The entries of

$$\left(\begin{array}{c} \mathrm{M}_k(\mathrm{A},\mathrm{A},\mathrm{A},\mathrm{A}) \\ \mathrm{M}_k(\mathrm{A}^2,\mathrm{A},\mathrm{A},\mathrm{I}) \\ \mathrm{M}_k(\mathrm{A}^2,\mathrm{A}^2,\mathrm{I},\mathrm{I}) \\ \mathrm{M}_k(\mathrm{A}^3,\mathrm{A},\mathrm{I},\mathrm{I}) \\ \mathrm{M}_k(\mathrm{A}^4,\mathrm{I},\mathrm{I},\mathrm{I}) \end{array}\right) = \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & -1 & 4 \\ 1 & -4 & 2 & 4 & -4 \end{array}\right) \left(\begin{array}{c} \mathrm{A}^{(n,k,1)4} \\ \mathrm{A}^{(n,k,1)2}\mathrm{A}^{(n,k,2)} \\ \mathrm{A}^{(n,k,2)2} \\ \mathrm{A}^{(n,k,1)}\mathrm{A}^{(n,k,3)} \\ \mathrm{A}^{(n,k,4)} \end{array}\right), \tag{106}$$

are similarly assembled working from top to bottom.

4.3 Eigenvalue considerations

Proposition 7. If A is a symmetric $n \times n$ matrix, the matrix $\mathrm{A}^{(n,k,s)}$ has the eigenvalue $a_s(\lambda_R)$, $s = 1,\ldots,k$, for each $R = (r_1,\ldots,r_k) \in Q_{k,n}$, where $a_s(\lambda_R)$ is the s-th ESF of $(\lambda_{r_1},\ldots,\lambda_{r_k})$.

Proof. Proposition 1 states that $C_k(\mathrm{A})$ has one eigenvector $\phi(e^R)$ with eigenvalue

$$\lambda_R = \lambda_{r_1}\ldots\lambda_{r_k} \tag{107}$$

for each $R = (r_1,\ldots,r_k) \in Q_{k,n}$. The matrix $C_k(\mathrm{I}+t\mathrm{A})$ and all of the $\mathrm{A}^{(n,k,s)}$ clearly possess the same eigenvectors as $C_k(\mathrm{A})$ itself. Hence, for each $R \in Q_{k,n}$, $C_k(\mathrm{I}+t\mathrm{A})$ has the eigenvalue

$$\prod_{j=1}^{k}(1+t\lambda_{r_j}) = \sum_{s=0}^{k} t^s a_s(\lambda_R), \tag{108}$$

and the result of the Proposition follows from (9).

Proposition 7 allows proof of a generalisation of the result given as $(xviii)$ of Section 7.5.17 of [4] for $s = 1$:

Proposition 8.

$$\mathrm{tr}\ \mathrm{A}^{(n,k,s)} = \left(\begin{array}{c} n-s \\ k-s \end{array}\right) a_s, \quad s = 0,\ldots,k. \tag{109}$$

Proof. The s-th term of the left side of (108) involves $\left(\begin{array}{c} k \\ s \end{array}\right)$ s-fold products of distinct eiegenvalues of A. Hence tr $\mathrm{A}^{(n,k,s)}$ involves $\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} k \\ s \end{array}\right)$ such products, which, by symmetry, cover the $\left(\begin{array}{c} n \\ s \end{array}\right)$ terms of $a_s = a_s(\lambda_1, \ldots, \lambda_n)$ a number of times equal to

$$\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} k \\ s \end{array}\right) / \left(\begin{array}{c} n \\ s \end{array}\right) = \left(\begin{array}{c} n-s \\ k-s \end{array}\right). \tag{110}$$

□

For $k = n-1$ and $k = n-2$, results more convenient for CE work can be deduced from Proposition 7.

Proposition 9. If A is a symmetric $n \times n$ matrix, and $k = n-1$, the eigenvalues of $\mathrm{A}^{(n,k,s)}$ can be given in the form

$$\sum_{j=0}^{s}(-\lambda_i)^j a_{s-j}, \quad i = 1, \ldots, n. \tag{111}$$

Proof. $\mathrm{A}^{(n,k,s)}$ has one eigenvalue $a_s(\lambda_R)$ for each $R = (r_1, \ldots, r_k) \in Q_{k,n}$. Let i be the element of the set $(1, \ldots, n)$ which is not a subscript of R. Then

$$a_s = a_s(\lambda_R) + \lambda_i a_{s-1}(\lambda_R). \tag{112}$$

Indeed

$$a_{s-j} = a_{s-j}(\lambda_R) + \lambda_i a_{s-j-1}(\lambda_R). \tag{113}$$

Multiply (113) by $(-\lambda_i)^j$, and sum over equations (113) from $j = 0$ to $j = s-1$. This yields

$$\sum_{j=0}^{s-1}(-\lambda_i)^j a_{s-j} = a_s(\lambda_R) - (-\lambda_i)^s a_0(\lambda_R). \tag{114}$$

But $a_0 = a_0(\lambda_R) = 1$, and so the result (111) of Proposition 9 follows. □

Remark 4. Eq. (111) yields

$$\mathrm{tr}\ \mathrm{A}^{(n,n-1,s)} = \sum_{j=1}^{s}(-1)^j S_j a_{j-s} + n a_s = (n-s) a_s, \tag{115}$$

in agreement with (109).

Proposition 10. If A is a symmetric $n \times n$ matrix, and $k = n-1$, $\mathrm{A}^{(n,n-1,s)}$ has one eigenvalue for each $i \in (1, \ldots, n)$, these values being the complement of some $R \in Q_{n-1,n}$. For $s = 1, 2, 3$, etc., this eigenvalue is given by

$$\begin{aligned} &a_1 - \lambda_i \\ &a_2 - \lambda_i(a_1 - \lambda_i) \\ &a_3 - \lambda_i\left[a_2 - \lambda_i(a_1 - \lambda_i)\right]. \end{aligned} \tag{116}$$

In the case of $\mathrm{A}^{(n,n-2,s)}$, there is one eigenvalue for each pair $(ij) \in Q_{2,n}$, the complement in $(1, \ldots, n)$ of $R \in Q_{n-2,n}$. For $s = 1, 2, 3$, etc., this eigenvalue is given by

$$\begin{aligned} &a_1 - (\lambda_i + \lambda_j) \\ &a_2 - (\lambda_i + \lambda_j)a_1 + ({\lambda_i}^2 + \lambda_i\lambda_j + {\lambda_j}^2) \\ &a_3 - (\lambda_i + \lambda_j)a_2 + ({\lambda_i}^2 + \lambda_i\lambda_j + {\lambda_j}^2)a_1 - ({\lambda_i}^3 + {\lambda_i}^2\lambda_j + \lambda_i{\lambda_j}^2 + {\lambda_j}^3), \end{aligned} \tag{117}$$

whilst for general s there is the expression

$$\sum_{m=0}^{s}(-1)^m h_m(\lambda_i,\lambda_j)a_{s-m}, \tag{118}$$

where h_j denotes the complete symmetric function defined by (iv) of Section 1.1.

Proof. A proof of the general result valid for all $s = 1,\ldots,k$, is not included, but the cases quoted can be easily checked to be correct, and extended if necessary. □

It is clear that all matrices of the type (13) have the same eigenvectors as the $C_k(\mathrm{A})$, so that there follows the valuable

Proposition 11. Let $R = (r_1,\ldots,r_k) \in Q_{k,n}$ and let $\phi(e^R)$ be the eigenvector of $C_k(\mathrm{A})$ belonging to the eigenvalue $\lambda_{r_1}\ldots\lambda_{r_k}$. Then the eigenvalue of the matrix of (13)

$$\mathrm{M}_k(\mathrm{A}^{s_1},\ldots\mathrm{A}^{s_k}) \tag{119}$$

to which $\phi(e^R)$ belongs is given by the MSF $M(s_1,\ldots,s_k)$ of the eigenvalues $\lambda_{r_1},\ldots,\lambda_{r_k}$ of A.

This allows the illustration of some general results alluded to in the introduction.

Example 9. Identities like those of Example 8 can be applied to the eigenvectors $\phi(e^R)$, $\quad R \in Q_{k,n}$, of $C_k(\mathrm{A})$ for $n\times n$ A. Eqs. (100, 103) and (105) provide identities involving various symmetric functions of the eigenvalues $\lambda_{r_1},\ldots,\lambda_{r_k}$ of A

$$\begin{aligned} {a_1}^2 &= S_2 + 2a_2 \\ a_1a_2 &= M(21) + 3a_3 \\ S_2a_1 &= S_3 + M(21),. \end{aligned} \tag{120}$$

All are easily seen to be true. Also they yield the well-known expansion

$$S_3 = S_2a_1 - S_1a_2 + 3a_3 = {a_1}^3 - 3a_1a_3 + 3a_3. \tag{121}$$

The results just quoted are contained also in the MSF analogue

$$\begin{pmatrix} M(1^3) \\ M(21) \\ M(3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 3 \end{pmatrix} \begin{pmatrix} a_1^3 \\ a_1a_2 \\ a_3 \end{pmatrix}. \tag{122}$$

of the MSM identity (104). A similar result can be easily written down in correspondence with (106), giving expressions for the $M(P_r)$, where P_r denotes a partition of 4, in terms of ESFs.

A general results Theorem 12 below of a different sort requires

Definition 10. Let $S_r(a_1,\ldots,a_r)$, for $r = 1,2,\ldots$, denote the unique explicit expression for the r-th power sum of some set of d indeterminates in terms of the ESFs of them.

Such expressions can be calculated [2] for increasing r values from the Newton relationships

$$\begin{aligned} S_1 - a_1 &= 0 \\ S_2 - S_1a_1 + 2a_2 &= 0 \\ S_3 - S_2a_1 + S_1a_2 - 3a_3 &= 0 \\ &\ldots \\ S_r - S_{r-1}a_1 + \ldots + (-1)^r ra_r &= 0, \end{aligned} \tag{123}$$

true for $r \leq d$. If $r > d$, the last term of S_r is $(-1)^d S_{r-d} a_d = 0$.

Proposition 12. The additive compound matrices $\Delta_k(\mathrm{A}^r)$ satisfy the formula

$$\Delta_k(\mathrm{A}^r) = S_r(\mathrm{A}^{(n,k,1)}, \ldots, \mathrm{A}^{(n,k,k)}), \tag{124}$$

with the qualification that terms of S_r in (124) corresponding to terms of $S_r(a_1, \ldots, a_n)$ involving arguments a_r with $r > k$ are absent in (124).

Proof. Let $R = (r_1, \ldots, r_k) \in Q_{k,n}$. Let $\phi(e^R)$ be the eigenvector of $C_k(\mathrm{A})$ with eigenvalue $\lambda_R = \lambda_1 \ldots \lambda_k)$. Then

$$\Delta_k(\mathrm{A}^r) = S_k(\lambda_R)\phi(e^R), \tag{125}$$

where

$$S_k(\lambda_R) = \sum_{j=1}^{k} {\lambda_{r_j}}^k, \tag{126}$$

and

$$\mathrm{A}^{(n,k,s)}\phi(e^R) = a_s(\lambda_R)\phi(e^R). \tag{127}$$

But

$$S_k(\lambda_R) = S_k(a_1(\lambda_R), \ldots, a_k(\lambda_R)), \tag{128}$$

by Definition 10. Since (127) applies to each one of a complete set of eigenvctors $\phi(e^R)$ as R varies over $Q_{k,n}$, the result of the Proposition follows. □

An inductive proof of (124) within the algebra of MSMs can be formulated.

The last lines of (101, 104) and (106) display results in agreement with Proposition 12 for $r = 2, 3, 4$.

5 Characteristic equations for the additive compound matrices $\Delta_k(\mathrm{A})$

Knowledge of the eigenvalues of $\mathrm{A}^{(n,k,s)}$ permits in principle the determination of its characteristic equation. A general means of accomplishing this has not been found, but a variety of systematic approaches and of interesting results are available. Consider first the important $s = 1$ case, easiest to treat for $k = n - 1$ and $k = 2$.

Proposition 13. If A is a symmetric $n \times n$ matrix, and $k = n-1$, the additive compound matrix $\Delta_{n-1}(\mathrm{A}) = \mathrm{A}^{(n,n-1,1)}$ has dimension n and characteristic equation

$$0 = \sum_{j=0}^{n} (\lambda - a_1)^{n-j} a_j. \tag{129}$$

Proof. The n distinct eigenvalues of $\Delta_k(\mathrm{A})$ are sums over $k = n - 1$ eigenvalues of A, and thus of the form

$$a_1 - \lambda_j, \quad j = 1, \ldots, n. \tag{130}$$

So $CE(\Delta_k(\mathrm{A}))$ is of the form

$$0 = \prod_{j=1}^{n} (\lambda - a_1 + \lambda_j), \tag{131}$$

and the required result follows. □

When $k = n-1$, use of (129) for $\lambda = 0$, gives rise to

$$\det \Delta_k(\mathrm{A}) = \det \mathrm{A}^{(n,n-1,1)} = \sum_{j=2}^{n} (-1)^j {a_1}^{n-j} a_j. \tag{132}$$

Proposition 14. If A is a symmetric $n \times n$ matrix, and $k = 2$, the additive compound matrix $\Delta_2(\mathrm{A}) = \mathrm{A}^{(n,2,1)}$ has dimension $\begin{pmatrix} n \\ 2 \end{pmatrix}$ and determinant

$$\det \Delta_2(\mathrm{A}) = \{n-1, n-2, \ldots, 1\}, \tag{133}$$

where the Schur function in (133) has also the interpretation as a $GL(n)$ character indicated in Sec. 1.1.

Proof. $\Delta_2(\mathrm{A})$ has one eigenvalue $(\lambda_i + \lambda_j)$ for each pair $(i, j) \in Q_{2,n}$, and so

$$\det \Delta_2(\mathrm{A}) = \prod (\lambda_i + \lambda_j) = \prod \frac{{\lambda_i}^2 - {\lambda_j}^2}{\lambda_i - \lambda_j} = \frac{\prod {\lambda_i}^2 - {\lambda_j}^2}{\prod \lambda_i - \lambda_j}, \tag{134}$$

where each product is over all pairs $(i, j) \in Q_{2,n}$. Eq. (134) thus shows that $\Delta_2(\mathrm{A})$ is the quotient of two Vandermonde determinants, and equal to the Schur function required. □

Remark 5. A formula expressing a Schur function in terms of the ESFs $a_1, \ldots, a_n$ is contained in Theorem 6 on page 96 of [1]. For $n = 3$, (133) gives rise to

$$\det \Delta_2(\mathrm{A}) = \{2, 1\} = \begin{vmatrix} a_2 & a_3 \\ 1 & a_1 \end{vmatrix} = a_1 a_2 - a_3 \tag{135}$$

in agreement with (132).

If $1 < k < n-1$ calculations are harder to complete and untidier in appearance, even for $\Delta_k(\mathrm{A})$. The simplest non-trivial case is that of

Example 10. If $k = 2$ and $n = 4$, $\Delta_2(\mathrm{A}) = \mathrm{A}^{(4,2,1)}$ has eigenvalues

$$\lambda_{i_1} + \lambda_{i_2}, \quad I = (i_1, i_2) \in Q_{2,4}. \tag{136}$$

Taking the factors of $\mathrm{CE}(\Delta_2(\mathrm{A}))$ in the obvious pairs, *e.g.*

$$(\lambda - \lambda_1 - \lambda_2)(\lambda - a_1 + \lambda_1 + \lambda_2) = X + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4), \tag{137}$$

where $X = \lambda(\lambda - a_1)$, it is a matter of routine calculation to show that $\mathrm{CE}(\Delta_2(\mathrm{A}))$ is a sextic equation in λ which can be written in the following concise form

$$0 = X^3 + 2a_2 X^2 + ({a_2}^2 + a_1 a_3 - 4a_4)X + (a_1 a_2 a_3 - {a_1}^2 a_4 - {a_3}^2). \tag{138}$$

When $n = 4$, so that a_5 is set equal to zero, Eq. (133) gives

$$\det \Delta_2(\mathrm{A}) = \{3, 2, 1\} = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_1 & a_2 & a_3 \\ 0 & 1 & a_1 \end{vmatrix} = a_1 a_2 a_3 - {a_1}^2 a_4 - {a_3}^2, \tag{139}$$

in agreement with (132).

More can be done for $\Delta_2(\mathrm{A})$ for $n \times n$ A, for which $\dim \Delta_2(\mathrm{A}) = N = \begin{pmatrix} n \\ 2 \end{pmatrix}$. Then the leading terms of $\mathrm{CE}(\Delta_2(\mathrm{A}))$ are found to be

$$\begin{aligned} 0 = \quad & \lambda^N - \begin{pmatrix} n-1 \\ 1 \end{pmatrix} a_1 \lambda^{N-1} + \left[\begin{pmatrix} n-1 \\ 2 \end{pmatrix} a_1{}^2 + (n-2)a_2 \right] \lambda^{N-2} \\ & - \left[\begin{pmatrix} n-1 \\ 3 \end{pmatrix} a_1{}^3 + (n-2)^2 a_1 a_2 + (n-4)a_3 \right] \lambda^{N-3} + \dots \, . \end{aligned} \tag{140}$$

Insertion of $n = 4$ into (140) allows agreement with the result (138) of Example 10 to be checked.

5.1 Characteristic equations for the matrices $\mathrm{A}^{(n,k,s)}$, $s > 1$

The cases of the matrices $\mathrm{A}^{(n,k,s)}$, in which $(n, k, s) = (4, 3, 2)$, $(5, 4, 2)$ and $(5, 4, 3)$, are treated.

Example 11. Proposition 10 implies that the matrix $\mathrm{A}^{(4,3,2)}$ has eigenvalues

$$a_2 - \lambda_j(a_1 - \lambda_j), \quad j = 1, \dots, 4. \tag{141}$$

Hence its CE is

$$0 = \prod_{j=1}^{4} [\lambda - a_2 + \lambda_j(a_1 - \lambda_j)] = \prod_{j=1}^{4} (\mu + c_j), \tag{142}$$

where $\mu = \lambda - a_2$ and $c_j = \lambda_j(a_1 - \lambda_j)$. To evaluate $\mathrm{CE}(\mathrm{A}^{(4,3,2)})$, use

$$\begin{aligned} \sum c_i &= a_1 M(1) - M(2) = a_1{}^2 - S_2 = 2a_2 \\ \sum c_i c_j &= a_1{}^2 M(11) - a_1 M(21) + M(22) = a_2{}^2 + a_1 a_3 + 2a_4 \\ \sum c_i c_j c_k &= a_1{}^3 M(111) - a_1{}^2 M(211) + a_1 M(221) - M(222) \\ &= a_1^2 a_4 + a_1 a_2 a_3 + 2a_2 a_4 - a_3{}^2 \\ c_1 c_2 c_3 c_4 &= a_4 \prod_{i=1}^{4} (a_1 - \lambda_i) = a_4 \det \mathrm{A}^{(4,3,1)} \\ &= a_4(a_1{}^2 a_2 - a_1 a_3 + a_4). \end{aligned} \tag{143}$$

The sums here are over $i, i < j, i < j < k$, and (178), with a_5, a_6 set to zero, has been used to evaluate the MSF involved. In the last line of (143), (132) for $n = 4$ has been used.

The CE (142) can now be written down as a quartic in μ and hence in λ. The most concise way of presenting is the one that stems from the method followed to derive it:

$$\begin{aligned} 0 = \quad & \lambda^2(\lambda - a_2)^2 + (a_1 a_3 + 2a_4)\lambda(\lambda - a_2) \\ & + (a_1{}^2 a_4 - a_3{}^2)\lambda + (a_4{}^2 - a_1 a_3 a_4 + a_2 a_3{}^2). \end{aligned} \tag{144}$$

Also

$$\begin{aligned} \det \mathrm{A}^{(4,3,2)} &= \prod_{j=1}^{4} (a_2 - c_j) = \prod_{j=1}^{4} (a_2 - \lambda_j a_1 + \lambda_j{}^2) \\ &= a_2{}^4 - a_2{}^3 \sum c_i + a_2{}^2 \sum c_i c_j - a_2 \sum c_i c_j c_k + c_1 c_2 c_3 c_4 \\ &= a_4{}^2 + a_2 a_3{}^2 - a_1 a_3 a_4. \end{aligned} \tag{145}$$

Example 12. The CE($\mathrm{A}^{(5,4,2)}$) follows with little more effort. The same forms as in (141) and (142) apply with $n = 5$ instead of $n = 4$ in three places. The first three results in (143) still apply, but, for $n = 5$, a_5 is kept in MSF formulas used. In the case present case, $(n, k, s) = (5, 4, 2)$, two additional results are needed

$$\begin{aligned}\sum c_i c_j c_k c_l &= {a_1}^4 M(1111) - {a_1}^3 M(2111) + {a_1}^2 M(2211) - a_1 M(2221) + M(2222) \\ c_1c_2c_3c_4c_5 &= a_5 \prod_{j=1}^{5}(a_1 - \lambda_j) = a_5 \det \mathrm{A}^{(5,4,1)} \\ &= a_5({a_1}^3 a_2 - {a_1}^2 a_3 + a_1 a_4 - a_5). \end{aligned} \tag{146}$$

The computer program prepared to express the MSFs involved here in terms of ESFs enables CE($\mathrm{A}^{(5,4,2)}$) to be obtained explicitly as a quintic first in μ and then λ. Thus, *cf.* (144), CE($\mathrm{A}^{(5,4,2)}$) is given as

$$\begin{aligned} 0 = \quad & \lambda^2(\lambda - a_2)^3 + (a_1a_3 + 2a_4)\lambda(\lambda - a_2)^2 \\ & +({a_1}^2 a_4 - {a_3}^2 + 3a_1a_5)\lambda(\lambda - a_2) \\ & \quad +({a_1}^3 a_5 - a_1a_3a_4 - 2a_3a_5 + a_2{a_3}^2 + {a_4}^2)\lambda \\ & \quad -({a_2}^2{a_3}^2 + a_2{a_4}^2 - a_1a_2a_3a_4 - 2a_2a_3a_5 + {a_1}^2a_3a_5 - a_1a_4a_5 + {a_5}^2). \end{aligned} \tag{147}$$

The expression within brackets in the last line gives an expression for the determinant term of CE($\mathrm{A}^{(5,4,2)}$)

$$\det \mathrm{A}^{(5,4,2)} = \prod_{j=1}^{5}(a_2 - \lambda_j a_1 + {\lambda_j}^2). \tag{148}$$

Example 13. The simplest nontrivial case in which $s = 3$ is that of the matrix $\mathrm{A}^{(5,4,3)}$. Proposition 7 implies that it has eigenvalues $a_3(\lambda_R)$ for each $R \in Q = Q_{4,5}$, *i.e.*,

$$R = (2, 3, 4, 5), (1, 3, 4, 5), (1, 2, 4, 5), (1, 2, 3, 5), (1, 2, 3, 4).$$

Proposition 10 shows that they can be given as

$$a_3 - a_2\lambda_i + a_1{\lambda_i}^2 - {\lambda_i}^3, \quad i = 1, \ldots, 5, \tag{149}$$

using the complements of $R \in Q$ as more convenient labels. The CE of $\mathrm{A}^{(5,4,3)}$ therefore takes the form

$$0 = \prod_{i=1}^{5}(\mu + b_i), \quad \mu = \lambda - a_3, \quad b_i = a_2\lambda_i - a_1\lambda_i^2 + {\lambda_i}^3. \tag{150}$$

To evaluate this explicitly the approach of the previous examples is followed

$$\begin{aligned} \sum b_i &= a_2M(1) - a_1M(2) + M(3) = 3a_2 \\ \sum b_ib_j &= {a_2}^2M(11) - a_1a_2M(21) + {a_1}^2M(22) + a_2M(31) - a_1M(32) + M(33) \\ \sum b_ib_jb_k &= {a_2}^3M(111) - {a_2}^2a_1M(211) + {a_1}^2a_2M(221) - {a_1}^3M(222) + {a_2}^2M(311) \\ & \quad -a_1a_2M(321) + {a_1}^2M(322) + a_2M(331) - a_1M(332) + M(333) \\ \sum b_ib_jb_kb_l &= {a_2}^4M(1111) - {a_2}^3a_1M(2111) + {a_2}^2{a_1}^2M(2211) - a_2{a_1}^3M(2221) \\ & \quad +{a_1}^4M(2222) + {a_2}^3M(3111) - a_1{a_2}^2M(3211) + {a_1}^2a_2M(3221) \\ & \quad -{a_1}^3M(3222) + {a_2}^2M(3311) - a_1a_2M(3321) + {a_1}^2M(3322) \\ & \quad +a_2M(3331) - a_1M(3332) + M(3333) \\ b_1b_2b_3b_4b_5 &= a_5 \det \mathrm{A}^{(5,4,2)}. \end{aligned} \tag{151}$$

Hence, as in Example 12 for $\mathrm{CE}(\mathrm{A}^{(5,4,2)})$, the $\mathrm{CE}(\mathrm{A}^{(5,4,3)})$ is obtained

$$\begin{aligned} 0 &= \lambda^5 - 2a_3\lambda^4 + (2a_1a_5 + a_2a_4 + {a_3}^2)\lambda^3 \\ &\quad +(-a_1{a_4}^2 - 2a_1a_3a_5 + {a_2}^2a_5 - a_2a_3a_4 - 3a_4a_5)\lambda^2 \\ &\quad +({a_1}^2{a_5}^2 - a_1a_2a_4a_5 - a_1a_3{a_4}^2 + {a_4}^3 + 3a_3a_4a_5 - 2a_2{a_5}^2)\lambda \\ &\quad -a_1a_4{a_5}^2 + a_2{a_4}^2a_5 - a_3{a_4}^3 + {a_5}^3. \end{aligned} \tag{152}$$

Remark 6. There are easy conjectures extending (143) and (151) for use, *e.g.*, in obtaining $\mathrm{CE}(\mathrm{A}^{(6,5,2)})$ or $\mathrm{CE}(\mathrm{A}^{(6,5,3)})$, in terms of MSFs with all labels less than or equal to 2 and 3. Labels 4 enter for $\mathrm{CE}(\mathrm{A}^{(6,5,4)})$.

Example 14. In order to exhibit a modest but convincing indication of the correctness of the calculations in the last two examples, which are in any case computer output, consider the matrix

$$\mathrm{A} = \mathrm{diag}\ (1, 2, 3, 4, 5), \tag{153}$$

for which the ESFs of its eigenvalues are

$$a_1 = 15, a_2 = 85, a_3 = 225, a_4 = 274, a_5 = 120. \tag{154}$$

Since $\mathrm{A}^{(5,4,2)}$ and $\mathrm{A}^{(5,4,3)}$ are the t^2 and t^3 terms in the expansion of $C_4(\mathrm{I}+t\mathrm{A})$, it is easy to verify that

$$\mathrm{A}^{(5,4,2)} \equiv \mathrm{diag}\ (71, 59, 49, 41, 35), \quad \text{and} \quad \mathrm{A}^{(5,4,3)} \equiv \mathrm{diag}\ (154, 107, 78, 61, 50), \tag{155}$$

and to write down their CEs. These agree with the results of inserting values from (154) into (147) and (152).

6 Compound matrices and representation theory

In this section it is assumed that A is hermitian, rather than merely real and symmetric. Most of what has been said elsewhere in this paper apply in this situation anayway. Further compound matrices $C_k(\mathrm{U})$, where U is an $n \times n$ unitary matrix, enter the studies. They too share with the matrices $C_k(\mathrm{A})$, for A hermitian, many of the properties listed above, such as the Binet-Cauchy theorem, (45).

Definition 11. Let g, $g \in \mathcal{G}$, denote an element of a compact Lie group $\mathcal{G}$. Let $g \mapsto \mathrm{U}(g)$ for each $g \in \mathcal{G}$ define the matrix $\Gamma(g)$ of a unitary representation Γ of dimension n of $\mathcal{G}$ in a vector space $\mathcal{V}$.

Proposition 15. The matrices $C_k(\mathrm{U})$, $\quad k = 1, \ldots, n$, are the matrices of the unitary representation $\bigwedge^{(k)}(\Gamma)$ of $\mathcal{G}$ acting in the vector space $\bigwedge^{(k)}(\mathcal{V})$.

Proof. That the matrices $C_k(\mathrm{U})$ describe endomorphisms of $\bigwedge^{(k)}(\mathcal{V})$ follows from what has already been said . The representation property

$$C_k(\mathrm{U}_1)C_k(\mathrm{U}_2) = C_k(\mathrm{U}_1\mathrm{U}_2), \tag{156}$$

for each pair of matrices $\quad \mathrm{U}_1, \mathrm{U}_2 \in \Gamma$ follows directly from the Binet-Cauchy theorem. Application of the same theorem to the unitary property $\mathrm{U}\mathrm{U}^\dagger = \mathrm{U}^\dagger\mathrm{U} = \mathrm{I}$ of $\mathrm{U} \in \Gamma$, as on page 94 of [5], yields the unitarity of $C_k(\mathrm{U})$. $\square$

Remark 7. If $\det \mathrm{U} = 1$, as for example for $\mathrm{U} \in SU(n)$, then $\det C_k(\mathrm{U}) = 1$ holds also.

Remark 8. The matrices $C_k(\mathrm{U}), \quad \mathrm{U} \in \mathcal{G}$, are examples of invariant matrices as defined on page 180 of [2], where it is observed that such matrices provide matrix representations of $\mathcal{G}$.

To make contact with the Lie algebra $l(\mathcal{G})$ of $\mathcal{G}$, write $\mathrm{U} \in \Gamma(g)$ in terms of hermitian A as

$$\mathrm{U} = e^{it\mathrm{A}}, \quad t \in \mathbb{R}. \tag{157}$$

Proposition 16. If A is a hermitian $n \times n$ matrix, then

$$C_k(e^{it\mathrm{A}}) = e^{it\Delta_k(\mathrm{A})}, \quad t \in \mathbb{R}, \tag{158}$$

holds for $k = 1, \ldots, n$.

Proof. Let $\phi(e^R),\ R = (r_i, \ldots, r_k) \in Q_{k,n}$ be an eigenvector of $C_k(\mathrm{A})$ and hence of $C_k(e^{it\mathrm{A}})$. Then

$$\begin{aligned} C_k(e^{it\mathrm{A}})\phi(e^R) &= \phi(e^{it\mathrm{A}}e^{(r_1)}, \ldots, e^{it\mathrm{A}}e^{(r_k)}) \\ &= e^{it\lambda_{r_1}} \ldots e^{it\lambda_{r_k}} \phi(e^R) \\ &= e^{ita_1(\lambda_R)}\phi(e^R) = e^{it\Delta_k(\mathrm{A})}\phi(e^R). \end{aligned} \tag{159}$$

Since this applies to each one of a complete set of eigenvectors $\phi(e^R)$ in which R varies over $Q_{k,n}$, the result of the Proposition follows. □

It has beeen seen that the compound matrices $C_k(\mathrm{U})$ for each $k = 1, \ldots, n$, carry unitary matrix representations of $\mathcal{G}$. Further Proposition 16 implies that the hermitian additive compound matrices $\Delta_k(\mathrm{A})$ play a similar role for $l(\mathcal{G})$. Explicitly this is given by the representation property

Proposition 17.

$$[\ \Delta_k(\mathrm{A}_1)\ ,\ \Delta_k(\mathrm{A}_2)\ ] = \Delta_k([\ \mathrm{A}_1\ ,\ \mathrm{A}_2\ ]). \tag{160}$$

Proof. It is a matter of straightforward use of the definition (91) of $\Delta_k(\mathrm{A})$ as it is expressed in (91) or (93), to verify the truth of (160). □

The expansion of $C_k(e^{it\mathrm{A}})$ in terms of MSMs of the matrix A follows from

Proposition 18. The following result is true:

$$\Delta_k(\mathrm{A})^r = \sum_{P_r} \frac{r!}{\prod_{j=1}^k r_j!} \mathrm{M}_k(\mathrm{A}^{r_1}, \ldots, \mathrm{A}^{r_k}), \tag{161}$$

where the sum is over the partitions of $P_r = \{r_1, \ldots, r_k\}$ of r where $r_1 geq r_2 \geq \ldots \geq r_k$.

Proof. This is proved most easily on the basis of eigenvalue considerations since the result

$$(\lambda_{r_1} + \ldots \lambda_{r_k})^r = \sum_{P_r} \frac{r!}{\prod_{j=1}^k r_j!} M(r_1, \ldots, r_k), \tag{162}$$

involving MSFs of the $\lambda_{r_1}, \ldots, \lambda_{r_k}$, is proved, using a slightly different notation, as Theorem 1.13 of [17]. □

It can readily be checked that (161) yields results which agree with a direct calculation of the expansion of $C_k(e^{it\mathrm{A}})$ at least as far as its $r = 5$ term.

7 Difference equations

Definition 12. Let X_n denote the set $X_n = \{x_r, \quad r \in \mathbb{N}, \}$ of discrete variables which satisfy the difference equation DE(X_n)

$$0 = x_{n+r} - a_1 x_{n+r-1} + a_2 x_{n+r-2} - \ldots + (-1)^n a_n x_r, \tag{163}$$

together with initial conditions which specify the the values of

$$x_0, x_1, \ldots, x_{n-1}. \tag{164}$$

This DE contains exactly the same coefficients as occur in the CE(A): $0 = \prod_{j=1}^{n}(\lambda - \lambda_j)$. It follows that (163) has solutions of the form

$$x_r = \sum_{i=1}^{n} c_i {\lambda_i}^r, \tag{165}$$

where the constants $c_i, \quad i = 1, \ldots, n$, can be determined, if need be, via initial conditions for X_n.

This immediately prompts the question: what sets of discrete variables satisfy DEs bearing the same relationship, for $k = 1, \ldots, n$, to CE(C_k(A)) as X_n bears to that of A. Since the CE(C_k(A)) can written in the form

$$0 = \prod_{R \in Q_{k,n}} (\lambda - \lambda_R), \tag{166}$$

where $R = (r_i, \ldots, r_k)$ and $\lambda_R = \lambda_{r_1} \ldots \lambda_{r_k}$, this implies that the DEs sought are ones with solutions of the form

$$\sum_{R \in Q_{k,n}} d_R \lambda_R^r. \tag{167}$$

However variables for which this is true are already known [12].

Definition 13. Let H_s^r be the $s \times s$ Hankel matrix, $s = 1, 2, \ldots$, with elements

$$[H_s^r]_{ij} = x_{r+i+j-2}, \quad 1 \le i, j \le s, \quad r = 0, 1, \ldots, \tag{168}$$

and define the Hankel variable

$$h_s^r = \det H_s^r. \tag{169}$$

In this context the following notation

$$h_s^r = |x_r x_{r+1} \ldots x_{r+s-1}| \tag{170}$$

is useful. Here the first row only is displayed and remaining rows are constructed by systematically placing the entry x_{p+1} of row $(q+1)$ directly below the entry x_p of row q. Thus

$$\begin{aligned} h_1^r &= x_r \\ h_2^r &= y_r = |x_r x_{r+1}| = x_r x_{r+2} - {x_{r+1}}^2 && (171) \\ h_3^r &= z_r = |x_r x_{r+1} x_{r+2}| = \begin{vmatrix} x_r & x_{r+1} & x_{r+2} \\ x_{r+1} & x_{r+2} & x_{r+3} \\ x_{r+2} & x_{r+3} & x_{r+4} \end{vmatrix} && (172) \\ h_4^r &= w_r = |x_r x_{r+1} x_{r+2} x_{r+3}|, \quad \text{etc.} && (173) \end{aligned}$$

Also $h_0^r = 1$ and $h_{-1}^r = 0$ for all r, and as a result of (163), $h_{n+1}^r = 0$ for all r.

Proposition 19. The variable h_k^r for each fixed value of $k = 1, \ldots, n$, can be expanded in the form (167).

Proof. This is proved in [12] as Theorem 11 there, with the additional information that

$$d_R = \prod_{j=1}^{k} c_{r_j} v(R)^2, \tag{174}$$

where $v(R)$ is the Vandermonde determinant

$$v(\lambda_{r_1}, \ldots, \lambda_{r_k}) = \prod_{1 \le r_j < r_i \le k} (\lambda_{r_j} - \lambda_{r_i}). \tag{175}$$

□

Remark 9. In the case of $n = 4$, appplications of (163) to the definition $y_r = x_r x_{r+2} - {x_{r+1}}^2$ of (171) for y_r shows explicitly and without reference to eigenvalues that the y_r satisfy a DE with exactly the same coefficients as found in (76) of Example 3. Similarly, still for $n = 4$, the variables z_r defined by (172) are found to satisfy the expected analogue

$$0 = z_{r+4} - a_3 z_{r+3} + a_2 a_4 z_{r+2} - a_1 {a_4}^2 z_{r+1} + {a_4}^3 z_r, \tag{176}$$

of (81) for $C_3(\mathrm{A})$. Further, for $n = 5$ it can be shown, on the basis of (163) alone that y_r, z_r, w_r as defined in (171–173) obey DEs which agree coefficientwise, as (163) does with CE(A), with CE(C_k(A)) for $k = 2, 3, 4$.

Eq. (82) was first derived directly using (163) for $n = 5$. However the approach that uses eigenvalue considerations is much easier.

7.1 Use of X_n in relation to the QR algorithm for A

First note that the set X_n is fully determined to within a scale factor $x_0 = \gamma^2 \in \mathbb{R}$ by $2n - 1$ independent quantities: the $a_s, \quad s = 1, \ldots, n$ of (3) plus the initial values assigned to $x_1, \ldots, x_{n-1}$. Second turn to the case in which A is a real symmetric tridiagonal $n \times n$ matrix, so that it too involves $2n - 1$ independent quantities. These restrictions on A are not essential for the work of the present paper. But when they are imposed a mapping from tridiagonal A onto X_n can be realised by giving a rule for specifying $x_1, \ldots, x_{n-1}$ in terms of the elements of A. Such a mapping is an essential part of the discussion in [12]. In [12], explicit expressions are given, with proofs of their correctness, for all of the elements of all of the matrices

$$\mathrm{A}_r, \quad \mathrm{Q}_r, \quad \mathrm{R}_r, \quad r = 0, 1, \ldots, \tag{177}$$

of the QR algorithm [18–20] for $\mathrm{A} = \mathrm{A}_0$, an arbitrary real symmetric tridiagonal $n \times n$ matrix. The expressions in question are built entirely in terms of the variables $x_r \in X_n$. The Hankel variables of Definition 13 play a vital role results in presenting results in maximally concise form.

Appendix: Some results from the theory of symmetric functions

The following expressions for MSFs, required by the derivation of (143),

$$\begin{aligned}
M(21) = \sum {\lambda_1}^2 \lambda_2 \quad &= \quad \sum_{i<j} \lambda_i \lambda_j (\lambda_i + \lambda_j) = a_1 a_2 - 3a_3 \\
M(22) = \sum {\lambda_1}^2 {\lambda_2}^2 \quad &= \quad \sum_{i<j} {\lambda_i}^2 {\lambda_j}^2 = {a_2}^2 - 2a_1 a_3 + 2a_4 \\
M(211) = \sum {\lambda_1}^2 \lambda_2 \lambda_3 \quad &= \quad \sum_{i<j<k} \lambda_i \lambda_j \lambda_k (\lambda_i + \lambda_j + \lambda_k) = a_1 a_3 - 4a_4
\end{aligned}$$

$$M(221) = \sum {\lambda_1}^2{\lambda_2}^2\lambda_3 \quad = \quad \sum_{i<j<k} \lambda_i\lambda_j\lambda_k(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k) = a_2a_3 - 3a_1a_4 + 5a_5$$

$$M(222) = \sum {\lambda_1}^2{\lambda_2}^2{\lambda_3}^2 \quad = \quad \sum_{i<j<k} {\lambda_i}^2{\lambda_j}^2{\lambda_k}^2 = {a_3}^2 - 2a_2a_4 + 2a_1a_5 - 2a_6, \tag{178}$$

are true for all n. However, if *e.g.* $n = 5$, then a_6 must be set equal to zero.

Most of the results just quoted can be found explicitly in [1], where two methods of calculating particular MSFs are explained. It is straightforward to build on these results to derive the MSFs needed for Examples 11, 12 and 13. One example should suffice to show this goes.

Example 15. To treat $M(3311)$, for example, note the results

$$M(3311) \quad = \quad a_2M(33) - M(431) - M(44) \tag{179}$$

$$M(431) \quad = \quad a_1M(43) - M(53) - 2M(44), \tag{180}$$

and observe that all MSFs of the form $M(xy)$ are easily dealt with.

It may be worth remarking once more that the easiest way to find results such as (180) transfers attention to the corresponding MSM situation, and considers actions in $\bigwedge^{(4)}(\mathcal{V})$. Thus

$$M(\mathrm{A},\mathrm{A})M(\mathrm{A}^3,\mathrm{A}^3) = M(\mathrm{A}^3,\mathrm{A}^3,\mathrm{A},\mathrm{A}) + M(\mathrm{A}^4,\mathrm{A}^3,\mathrm{A}) + M(\mathrm{A}^4,\mathrm{A}^4), \tag{181}$$

and (179) follows eigenvalue considerations.

Tabulations of MSFs in terms of PSFs are available, see [21], noting the references cited there. The results can be written in terms of ESFs, but it is easier to proceed directly.

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