The Morley Tricorn: a solid figure constructed using the diagram for Morley's theorem as its net

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1 Introduction

Morley's theorem states that the points of intersection of adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle, known as Morley's triangle. This can be seen in Figure 1, where the angles at A, B, C are $69, 51, 60$ degrees.

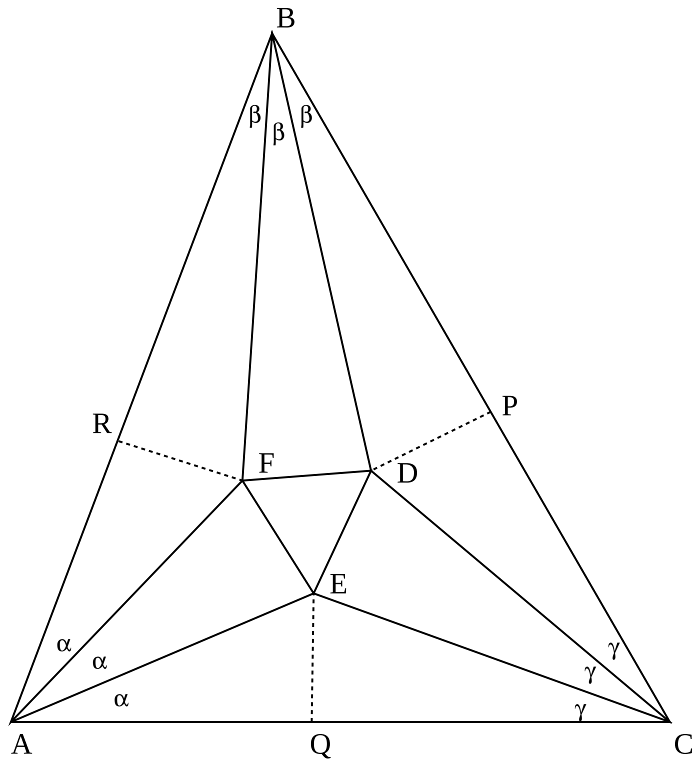

FIGURE 1

Concise proofs of the theorem are given in recent papers [1], [2]. A good picture of important previous work can be obtained by looking at [3] p1999, and examining references cited there.

The purpose of the present work is to demonstrate how the triangle shown in Figure 1 can be regarded as the 'net' for the construction of a unique rigid exactly closed solid with undistorted plane faces. The word net is here used in the same sense that [4] uses it in the construction of polyhedra.

To implement the contruction , it is necessary to define points P, Q, R on the sides BC, CA, AB of ΔABC such that

$$AQ = AR, \quad BR = BP, \quad CP = CQ. \tag{1}$$

These points are uniquely defined, giving line segments of lengths

$$AQ = s - a, \quad BR = s - b, \quad CP = s - c, \tag{2}$$

where $s = (a + b + c)/2$, in standard notation.

The solid is here called the Morley tricorn: see the entry for tricorn in *e.g.* [5]. It is easily described in qualitative term. Take a pyramid on the Morley ΔDEF as base, with slant sides DP, EQ, CR, the points P, Q, R of the original triangle having been brought into coincidence as the vertex W of the pyramid. Then stellate its slant faces by pyramids with vertices A, B, C. The latter pyramids will sometimes be referred to as the horns H_A, H_B, H_C of the tricorn at A, B, C.

It may help clarify both the geometrical nature of the tricorn as well as the mathematical proof of its existence (Sec. 3) to proceed first to the construction (Sec. 2) of a suitable model.

But first some notation is needed. Much of the ensuing discussion depends upon consideration of Figure 1 with cuts made along the lines DP, EQ, FR. Thus AC for example is cut into two parts with AQ_A on the same side of EQ as A, and CQ_C on the same side as C. Similar meanings are attached to the points R_A, R_B on AB, and the points P_B, P_C on BC, so that (1) and (2) give

$$AQ_A = AR_A = s - a, \quad \text{etc.} \tag{3}$$

2 Construction of a model of a tricorn

This begins with an accurate drawing like that of Figure 1 but as large as possible on an $A3$ sheet of hard card. Pin-pricks at the vertices allow the lines DP, EQ, CR to be copied onto the lower side of the $A3$ sheet. Gluing flaps should also be attached to AR, BP, CQ so as to be ready to tuck neatly, for eventual gluing, under triangles AEQ, BFR, CDP in the formation of the horns of the tricorn of A, B, C. Solid lines on the top side of the sheet are to receive valley-folds, whereas on dashed lines – only DP, EQ, CR – mountain folds are called for. Here the language of Origami is used [6]; see also [7], where much useful advice on folds and model building is to be found. To expedite the production of folds, solid lines should be scored with a hard sharp point on the top side of the sheet, and dashed lines scored on the lower side. Only light scoring on DE, EF, FD is needed. Crisp folds, in which special care is taken at the points that are to be the vertices of the Morley tricorn, are needed to get a nice model. Finally ΔABC with suitable flaps should be cut out.

While the actual folding of the final model is not at all difficult, some experience of Origami [6] or polyhedron building [4] is obviously a great help. Turn first to the formation of the horn H_A, with ΔABC cut as described at the end of Sec.1. Consider then the portion $AQ_A\,EFR_A\,A$ of Figure 1 together with the flap attached to AR_A. Valley folds along AE and AF bring triangles AFR_A and EAQ_A up over triangle EAF, taking the lines AR_A and AQ_A into coincidence as the line AW_A, Q_A and R_A having coalesced as W_A. In the process, it can be seen that, if the flap attached to AR_A has an angle at A somewhat smaller than α, it will tuck neatly under triangle AEQ_A, yielding a pyramid $EFW_A\,A$, firm even without gluing.

The language of rotations may be preferred. Since, from (3), AQ_A and AR_A are of equal length, it is sufficient to consider unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in these directions, see Figure 1, and to ask: for what value of ϕ do rotations of angles ϕ about AE and $-\phi$ about AF bring the tips of these unit vectors to the same end point? To answer it, recall the well-known formula

$$\mathbf{u}' = \cos\theta\;\mathbf{u} + (1-\cos\theta)\;\mathbf{n}.\mathbf{u}\;\mathbf{n} + \sin\theta\;\mathbf{n}\;\wedge\mathbf{u}, \tag{4}$$

for the rotation of angle θ about a unit vector $\mathbf{n}$ that carries $\mathbf{u}$ to $\mathbf{u}'$. Use Cartesian axes with x-axis along AC and z-axis up out of the plane of Figure 1, and the usual orthogonal set of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Take

$$\mathbf{u}_1 = \mathbf{i}, \quad \mathbf{n}_1 = \cos\alpha\;\mathbf{i} + \sin\alpha\;\mathbf{j},$$

for the rotation about AE, and for that about AF

$$\mathbf{u}_2 = \cos A\;\mathbf{i} + \sin A\;\mathbf{j}, \quad \mathbf{n}_2 = \cos 2\alpha\;\mathbf{i} + \sin 2\alpha\;\mathbf{j}.$$

Evaluating $\mathbf{u}_1{}'$ and $\mathbf{u}_2{}'$, it is found that $\mathbf{u}_1{}' = \mathbf{u}_2{}'$ when ϕ is given by

$$\cos\phi = -\frac{\cos\alpha}{1+\cos\alpha}.$$

For this value of ϕ then AQ_A and AR_A may be brought into coincidence.

Going back now to the actual construction of a model of the Morley tricorn from an uncut sheet of card, it is soon seen that the process of loosely forming the three horns leads naturally to making the valley folds along DP, EQ, FR, *as seen from the lower side of the card*, needed to bring P, Q, R into exact coincidence at unique point W. The final stage of the process works very easily: the tricorn can be held with everything in place for inspection, ahead of possible gluing.

Thus is realised the Morley tricorn, with central pyramid $DEFW$ with base the Morley triangle and vertex W, whose slant sides EFW, FDW, DEW are stellated with the horns or pyramids $H_A = EFWA, \quad H_B = FDWB, \quad H_C = DEWC$.

3 Proof of existence

For the analysis of the existence of the Morley tricorn, it again is convenient to consider ΔABC with cuts along DP, EQ, FR. In the actual folding process of course these lines do not need to be cut. But the cutting has already allowed consideration of the formation of each of the pyramids H_A, H_B, H_C in isolation.

Having completed construction of the three horns, their assembly into the tricorn remains to be considered. The three pyramids just mentioned have base outlines in the form of triangles EFW_A, FDW_B, DEW_C. Focus on the context in which these three triangles are mountain folded downwards at EF, FD, DE as needed to make them lie in the plane of the Morley triangle ΔABC. In other words, the whole horn at A, H_A, (*e.g.*), should be rigidly rotated about EF until its base triangle EFW_A lies in the plane of the Morley triangle. It is obvious that this can be done. See Figure 2, which is accurate when ΔABC has angles $102, 60, 18$ degrees. In virtue of the cuts, and of rotations made in forming the horns of the tricorn, Figure 2 reflects these equalities of length

$$\begin{aligned} DW_C &= DP_C = DP_B = DW_B \\ EW_C &= EQ_C = EQ_A = EW_A \\ FW_A &= FR_A = FR_B = FW_B. \end{aligned} \tag{5}$$

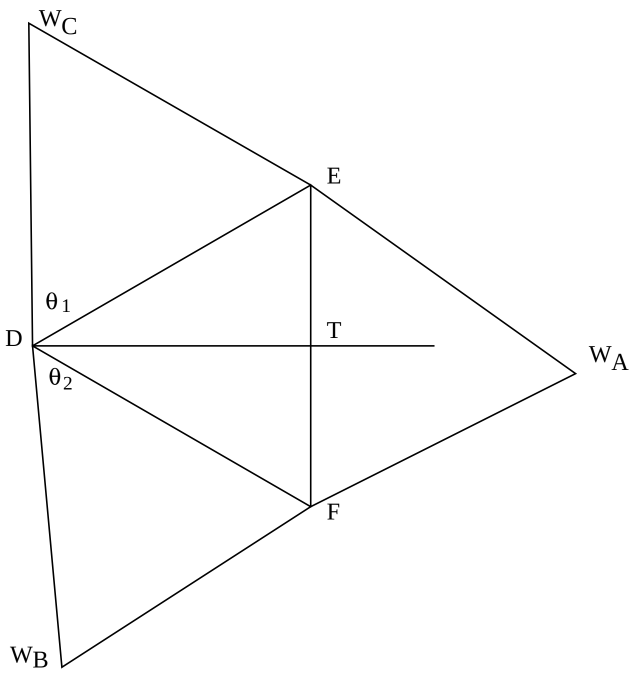

FIGURE 2

To establish closure of the three base triangles to form the pyramid $DEFW$, with W_A, W_B, W_C brought into coincidence at W, and hence to prove the existence of the Morley tricorn, proceed in two stages.

In Figure 2, take the origin at D, the x-axis along DT, where T is the mid-point of EF, and z-axis up and out of the plane of the diagram. The key question is: can rotations ϕ_1 about DE and $-\phi_2$ about DF be chosen to bring DW_C and DW_B into coincidence as DW_{BC}? Because of the first part of (5), it is sufficient to consider unit vectors $\mathbf{u}_1$ along DW_C, and $\mathbf{u}_2$ along DW_B, and to satisfy the requirement $\mathbf{u}_1{}' = \mathbf{u}_2{}'$. Here

$$\mathbf{u}_1 = \cos(\theta_1 + 30)\,\mathbf{i} + \sin(\theta_1 + 30)\,\mathbf{j}, \quad \mathbf{n}_1 = \cos 30\,\mathbf{i} + \sin 30\,\mathbf{j},$$

$$\mathbf{u}_2 = \cos(\theta_2 + 30)\,\mathbf{i} - \sin(\theta_2 + 30)\,\mathbf{j}, \quad \mathbf{n}_2 = \cos 30\,\mathbf{i} - \sin 30\,\mathbf{j}.$$

It may be verified by routine manipulations, based on (4) and familiar trigonometric identities, that

$$\mathbf{i}.\mathbf{u}_1{}' = \mathbf{i}.\mathbf{u}_2{}', \quad \mathbf{j}.\mathbf{u}_1{}' = \mathbf{j}.\mathbf{u}_2{}',$$

are both satisfied when

$$\cos\phi_1 = \frac{\cos\theta_1 - 2\cos\theta_2}{\sqrt{3}\sin\theta_1}, \quad \cos\phi_2 = \frac{\cos\theta_2 - 2\cos\theta_1}{\sqrt{3}\sin\theta_2}. \tag{6}$$

These results imply that

$$\sin^2\phi_1 \sin^2\theta_1 = \sin^2\phi_1 \sin^2\theta_1. \tag{7}$$

Since all the angles involved are less than π and hence have positive sines, (7) gives

$$\sin\phi_1 \sin\theta_1 = \sin\phi_1 \sin\theta_1.$$

This ensures that $\mathbf{k}.\mathbf{u}_1{}' = \mathbf{k}.\mathbf{u}_2{}'$ so that the answer yes to the key question has been obtained.

It is to be noted that $\cos\phi_1$ and $\cos\phi_2$ can be evaluated from (6) in principle for any given values of θ_1 and θ_2. This is fortunate because the expressions in (2) make use of the cosine rule to evaluate $EQ = EQ_C = EW_C$ etc., very awkward. It follows that viable explicit expressions for the angles θ_1 and θ_2 seen in Figure 2 are likely to remain unavailable. Thus it should be asked whether, in all cases, (6) gives well-defined angles ϕ_1 and ϕ_2. Strong evidence, rather than proof, that indeed it does so is provided in an appendix.

It has been seen that the triangles DEW_C and FDW_B can be rotated carrying the horns H_C and H_B 'behind' them, so as to bring DW_C and DW_B into coincidence, with W_C and W_B coalescing as W_{BC}. This process has defined two slant sides of a pyramid $DEFW_{BC}$. It is 'open' at the third face EFW_{BC}, a triangle with sides EF, EW_{BC}, FW_{BC}. But the following identifications of lengths, see also (5), are valid

$$EW_{BC} = EW_C = EW_A, \quad FW_{BC} = FW_B = FW_A.$$

Hence triangle EFW_{BC} and triangle EFW_A, with corresponding sides equal, are congruent. Hence it is clear that the latter can be rotated up about EF to fit exactly into the opening left at the end of the previous stage of the analysis. At the same time the horn H_A is carried into its final position in the tricorn, whose existence has been demonstrated.

The construction described here of the Morley tricorn was first presented at the Symposium Quantum Groups, Foundations, Information

celebrating the work of Tony Sudbery, which was held at the University of York, 29-30 September 2008, upon his retirement from his Professorship in the Department of Mathematics there. A logo of this department features the diagram for Morley's theorem.

Comment on the angles defined by equation (6)

A computer program, which uses formulas from [1] and the sine and cosine rules, shows that, for a wide variety of triangles ABC, including cases such as those with angles $15, 21, 144$ and $160, 10, 10$, the triangles DEW_C, EFW_A, FDW_B all turn to be *amazingly* close to being equilateral, though they never are even when ΔABC itself is. See also Figure 2, where *all* the angles lie between 54.5 and 63.4 degrees. Thus, from (6), the angles ϕ_1 and ϕ_2 may be expected to lie close to $\arccos(1/3) \approx 109.5$ degrees, and so to be well-defined. Otherwise note that $|\cos\phi_1| \leq 1$, and $|\cos\phi_2| \leq 1$ each require that the angles θ_1 and θ_2 satisfy

$$\cos^2\theta_1 - \cos\theta_1 \cos\theta_2 + \cos^2\theta_2 \leq 3/4, \tag{8}$$

and that the left side of (8) is equal to $1/4$ for angles $\theta_1 = \theta_2 = 60$ degrees. While remarks just made and detailed numerical work indicate that use of (6) is problem-free, the lack of formulas for angles like θ_1 and θ_2 means that formal proof that this is so remains to be completed.

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