On the Steiner deltoid, the twelve point circle and the Eulerian image triangle of an arbitrary plane triangle

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Abstract

By considering the Steiner deltoid of an arbitrary plane triangle ΔABC, that is, the envelope, as a point X of the circumcircle of ΔABC tranverses this circle, of the Simson line $S(X)$ of X in ΔABC, a clear pattern is seen in the points where the sides and altitudes of ΔABC touch the deltoid. From the geometry of this situation, there emerges, alongside the three familiar triples of points lying on the nine-point circle of ΔABC, a fourth triple, so that already a case exists for referring to the twelve-point circle of ΔABC. Examining the set of six concurrency and two non-concurrency theorems involving pairs of such triples strengthens the case, and leads to the introduction of a new triangle, in some sense dual to ΔABC. This yields the final picture in which is seen a simple two-fold symmetry Σ under rotation through 180 degrees about the centre of the mutual nine-point, now twelve-point, centre of the two triangles. Since under this rotation the circum and orthocentres of ΔABC become the ortho and circumcentres of the second triangle, the latter has been termed the Eulerian image of the former.

The role of the deltoid in the development of ideas described leads to the emergence of a new triangle centre. This fits well into the framework of the famous encyclopedia of triangle centres: it is the de Longchamps point of the orthic triangle of ΔABC.

Keywords: Simson line, Steiner deltoid, nine-point circle, twelve-point circle, Euler line, Eulerian image triangle, triangle centre, de Longchamps point.

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1 Introduction

This paper offers a serious attempt to justify the assertion: that, underlying the familiar geometry [1] of a general plane triangle – call it ΔABC – with its concurrency theorems, Euler line and nine-point circle, there exists a larger perhaps more obviously complete picture.

The motivation for thinking along these lines comes from the introduction of the Steiner hypocycloid, [2], or deltoid, [3] page 72, of ΔABC. This is the envelope [4] of the Simson line $S(X)$ [5], page 137, of a point X on the circumcircle of ΔABC, generated as X moves around the circumcircle [6]. The deltoid touches all the sides and altitudes of ΔABC, creating an elegant geometric pattern [7] of points of contact, involving all the points of the nine-point circle and a certain triple of others. This is not only the basis for arguing that the status of the nine-point circle merits promotion to that of twelve point circle, but also it leads directly and surely compellingly to the appearance of a second triangle in some sense dual to the given triangle ΔABC, as well as being congruent to it, and symmetrically related to it. As can be seen from Fig. 1, the symmetry Σ in question is a two-fold symmetry under a rotation of 180 degrees about N, the centre of the common nine-point circle of the two triangles. In view of the role of the Euler line of ΔABC in this symmetry, it is suggested that the second triangle be called the Eulerian image of ΔABC.

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Many of the geometric features of the promised larger picture, immediately plain to see in Fig. 1, are discussed in detail in the body of this paper. For further material regarding the deltoid, see, for example, [8–10].

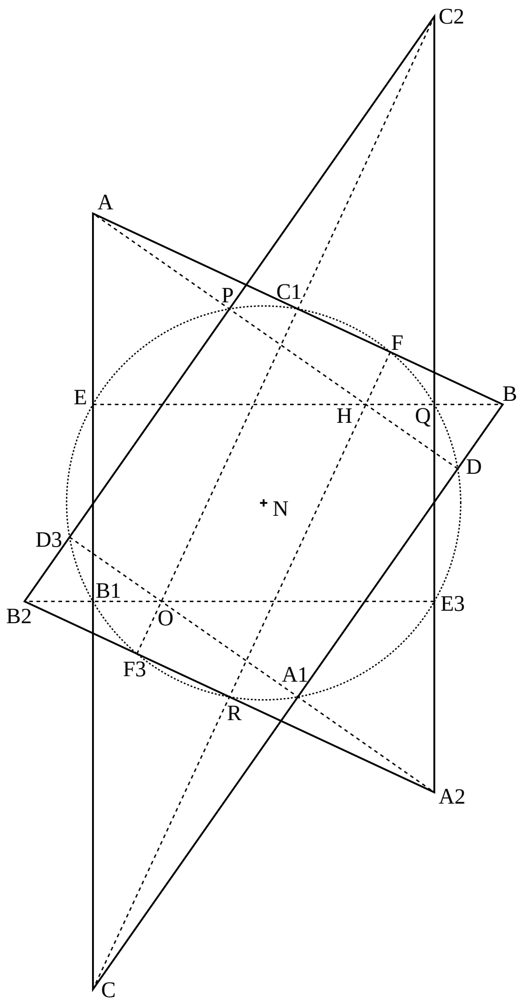

Fig. 1: ΔABC, the second or dual triangle $\Delta A2B2C2$, and the nine-point circle

In preparation for describing the content of the rest of this paper, it is useful to give a comprehensive listing of the letters used for the many triples of points to be dealt with. This includes

(i) the vertices A, B, C of ΔABC,

(ii) the midpoints $A1, B1, C1$ of the sides BC, CA, AB of lengths a, b, c of ΔABC,

(iii) the feet of D, E, F on the sides BC, CA, AB of ΔABC of its altitudes AD, BE, CF,

(iv) the midpoints P, Q, R of the segments AH, BH, CH of the altitudes AD, BE, CF, where H is the orthocentre of ΔABC.

The triples (ii) – (iv) are the set of nine-points which give the nine-point circle its name

The circumcircle of ΔABC has centre O and radius R, while the nine-point circle of ΔABC

has centre N, radius $\frac{1}{2}R$. The points H, O, N and the centroid G all lie on the Euler line $OGNH$ of ΔABC [1], page 27, [5], page 195. As becomes important later, in the late paragraphs of Section 6, so also does the de Longchamps point L [11–13] of ΔABC.

In view of their status as the Simson lines of certain known points of the circumcircle of ΔABC, its sides and altitudes are tangent to the deltoid. This brings into immediate focus, two more triples of points, seen in Fig. 2, none of which lie on the nine-point circle,

(v) the points $D1, E1, F1$ where the sides BC, CA, AB touch the deltoid,

(vi) the points $D2, E2, F2$ where the altitudes AD, BE, CF touch the deltoid.

Continuing, a fourth triple of points which do lie on the nine-point circle arises. This is provided by

(vii) the mid-points $D3, E3, F3$ of the lines $D1D2, E1E2, F1F2$, see Fig. 3. These lines are themselves of interest. In particular, see Section 9 for discussion of their point of concurrency K, a triangle centre not yet located in [14].

While the last of the main listings introduces the vertices

(viii) $A2, B2, C2$ of the Eulerian image of ΔABC, seen in Fig. 1, further triples (ix,x,xi) are mentioned below

A glance at the long list of points assembled, and any serious attempt to create diagrams that make clearly visible enough detail of the geometry of the contexts under study, will confirm the value of a good pictorial framework, and a good parametrisation of the co-ordinates of the points. The early paper [7] on the deltoid, and the various points of contact with it, provides an extremely convenient approach, which is followed here. Take the centre O of the circumcircle as origin, the x-axis along the perpendicular bisector $OB1$, and the y-axis along a line through O parallel to the side CA of ΔABC. Then, if the points A, B, C, which of course do lie on the circumcircle, are assigned co-ordinates according

$$A = R(\cos 2\alpha, \quad \sin 2\alpha), \quad B = R(\cos 6\beta, \quad \sin 6\beta), \quad C = R(\cos 2\alpha, \quad -\sin 2\alpha), \tag{1}$$

the angles of ΔABC

$$\angle A = \alpha + 3\beta, \quad \angle B = \pi - 2\alpha, \quad \angle C = \alpha - 3\beta, \tag{2}$$

and the co-ordinates of all other points of interest can written in terms of R and the two independent angles α, β. See Appendix A. This information can be used to write a computer program which enables choices of input α and β suitable for various illustrations.

All Figures appearing here use the values $\alpha = 50$ degrees and $\beta = 5$ degrees. In this case both the circumcentre O and the orthocentre H lie inside ΔABC, but for other choices of input this need not be so. See [5] paragraph 249 on page 161. Two features of the van Horn set-up may be noted. First, the discussion of features relating to CA and the altitude BE is always specially easy, allowing patterns to be seen that are somewhat harder to establish for the pair BC and AD, and for the pair AB and CF. Second the angle 6β used in (1) to specify the co-ordinates of B has been chosen to avoid the appearance of fractional angles, because 3β occurs in (2) and 2β in (7).

The content of the rest of this paper can be summarised as follows. Section 2 discusses the deltoid as the envelope of the Simson line $S(X)$. Section 3 uses Fig. 2 to describe the locations of the points where the sides and altitudes of ΔABC touch the deltoid, and Section 4 sketches the justifications of these results. Section 5, with the aid of Fig. 5, drawn from Fig. 2 for clarity, treats geometrical facts already made evident by Fig. 2. Section 6 contains the major findings of the paper. It discusses concurrency and non-concurrency issues involving the twelve points (ii), (iii), (iv) and (vii). The former provide another plank in the case for thinking of the nine-point circle as more naturally the twelve point circle. The latter bring the dual triangle $\Delta A2B2C2$ forcefully into consideration, completing the picture seen in Fig. 1, allowing a good body of the associated geomtrical results to be presented. Section 7, beginning with reference to the line $E1E2$, discusses the many instances of an interesting result [3], item 12, page 79, for the lengths of tangents to the deltoid. Section 8 collects in one place a review of the many concurrency theorems that are met with in the course of the present study, and indicates that trilinear co-ordinates offer a powerful general approach to them. In one case, that of the lines $AD1, BE1, CE1$, the point Z of concurrency is the known triangle centre $X(69)$ in [14]. In the case of the lines $D1D2$, $E1E2$ and $F1F2$ the Cartesian and trilinear co-ordinates of their point K of concurrency are determined. Although K has not so far been located in [14], there is a natural place for it in the framework of [14]. This in described in full detail in Section 9, which identifies the line KN with the generalised Euler line of the orthic [14] triangle ΔDEF, that is, the pedal triangle [5] of the orthocentre H of ΔABC in ΔABC. It is proved that the point K is the de Longchamps point of ΔDEF.

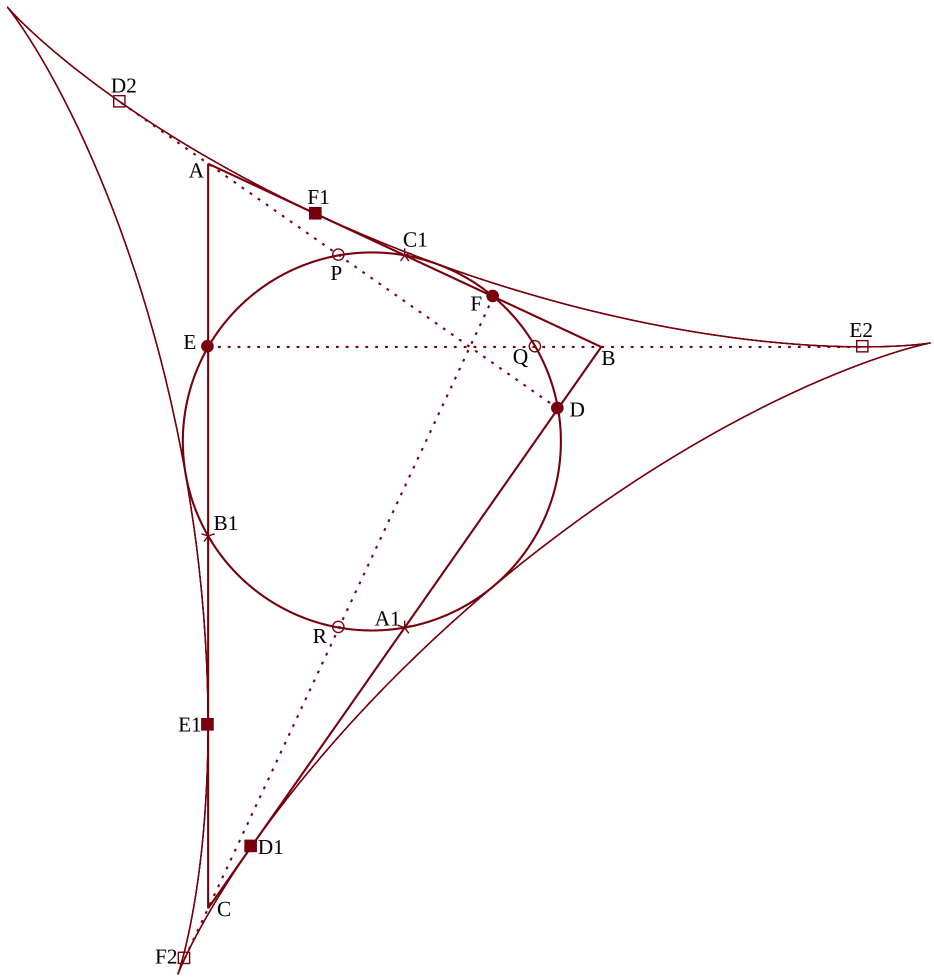

Fig. 2: The deltoid and the nine-point circle

Triples of points of interest are indicated in Fig. 2 by symbols.

For triples (ii) – (vi) the respective symbols are asterisks, solid-circles, open-circles, solid-squares, open squares.

By noting the positions of the open-squares for the points $D2, E2, F2$ where the altitudes of ΔABC touch the deltoid, it can be inferred that the cusps of the deltoid do not lie in general on the altitudes of ΔABC.

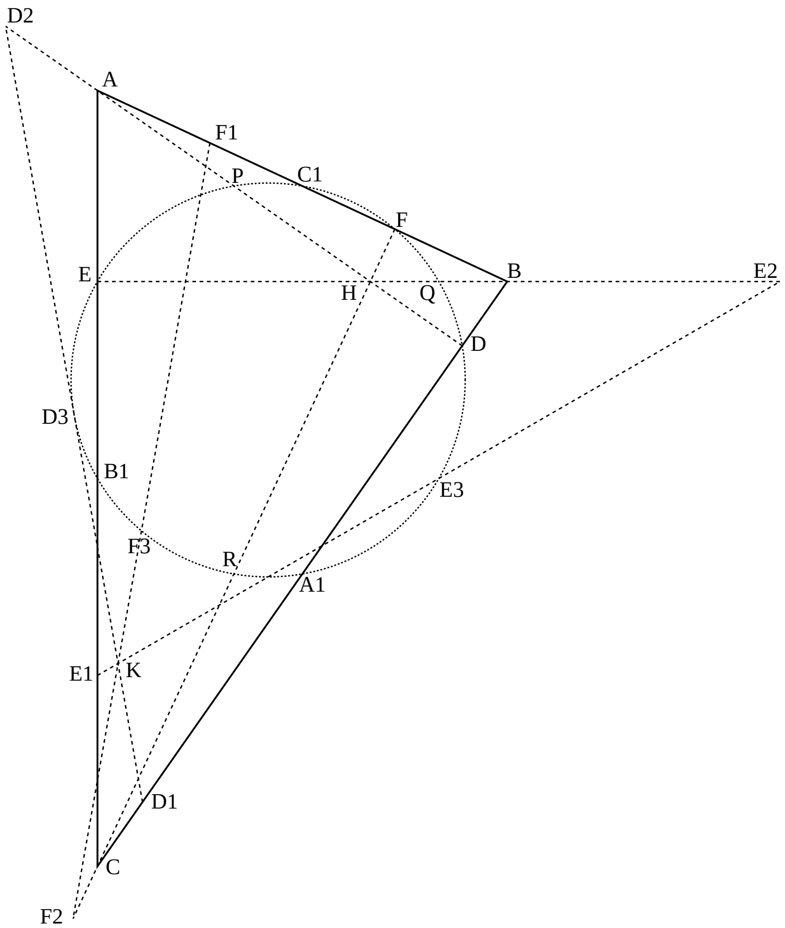

Fig. 3: All the points (i) – (vii)

Four appendices collect formulas for

(A) Co-ordinates of (almost) all the points in a van Horn [7] co-ordinate system with the parametrisation of angle used in [7].

(B) Equations of sides and altitudes of ΔABC and various other lines.

(C) Trilinear co-ordinates of many of the points of interest.

(D) Lengths of lines seen in the Figures.

2 The deltoid

The Simson line $S(X)$ for the point $X = R(\cos 2\phi, \quad \sin 2\phi)$ using the van Horn co-ordinate system at O has the equation

$$y - R\sin 2\phi = \tan(3\beta - \phi)(x - R\cos 2\alpha). \tag{3}$$

The envelope [4] of $S(X)$ as X moves round the circumcircle of ΔABC is given in parametric form, as in [7], as

$$\begin{aligned} x' = x - x_N &= R\cos 2\phi + \frac{R}{2}\cos(4\phi - 6\beta), \\ y' = y - y_N &= R\sin 2\phi - \frac{R}{2}\sin(4\phi - 6\beta). \end{aligned} \tag{4}$$

To see exactly how this relates to the geometry of ΔABC requires a clockwise rotation of angle 2β about N:

$$z' = x' + iy' \mapsto \zeta = \xi + i\eta = e^{-2i\beta}z', \tag{5}$$

so that

$$\xi = R(\cos 2\psi + \frac{1}{2}\cos 4\psi), \quad \eta = R(\sin 2\psi - \frac{1}{2}\sin 4\psi), \tag{6}$$

where $2\psi = 2\phi - 2\beta$. The deltoid described by such parametric equations is discussed extensively in [3], page 76. Its interpretation as the hypocycloid which arises as the locus of a fixed point on a circle of radius $\frac{R}{2}$ rolling round a fixed circle of the same radius is found there. The deltoid can be seen to be a curve with three-fold rotational symmetry about its centre N, and with three cusps $Cp1, Cp2, Cp3$. The lines $NCp1, NCp2, NCp3$ make angles $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ with the $\psi = 0$ direction at N, or angles

$$2\beta, 2\beta + \frac{2\pi}{3}, 2\beta + \frac{4\pi}{3} \tag{7}$$

with the x' direction at N. Also the cusps all lie at distance $\frac{3R}{2}$ from N. Obviously $\Delta Cp1Cp2Cp3$ is equilateral. This is is the second time that such a triangle has forced itself into view in the geometry of a general plane triangle, the first case of course being the Morley triangle [16–20], whose sides are antiparallel [3] to those of $\Delta Cp1Cp2Cp3$. A proof of this fact is offered in Section 2.1.

It may be worth remarking on a related observation that appears in [15], page 120, and on the language used in the context: it seems hardly credible that something as metrically precise as an isosceles right triangle should emerge (from a construction that sets out from an arbitrary plane triangle).

2.1 The antiparallel property of the sides of the two equilateral triangles

In this section and only in this section, the letters D, E, F, Q are the standard letters seen often in discussion of Morley's triangle [16], page 49. Thus here ΔDEF is Morley's triangle and Q is one of the vertices of the adjunct Morley triangle [16], ibid. Further the lines EV and EU are parallel to the x and y axes of the van Horn set-up. The aim is to prove the line EQ in Fig. 4 makes an angle 2β with the horizontal, just as the line $NCp1$ has been shown to do, see (7).

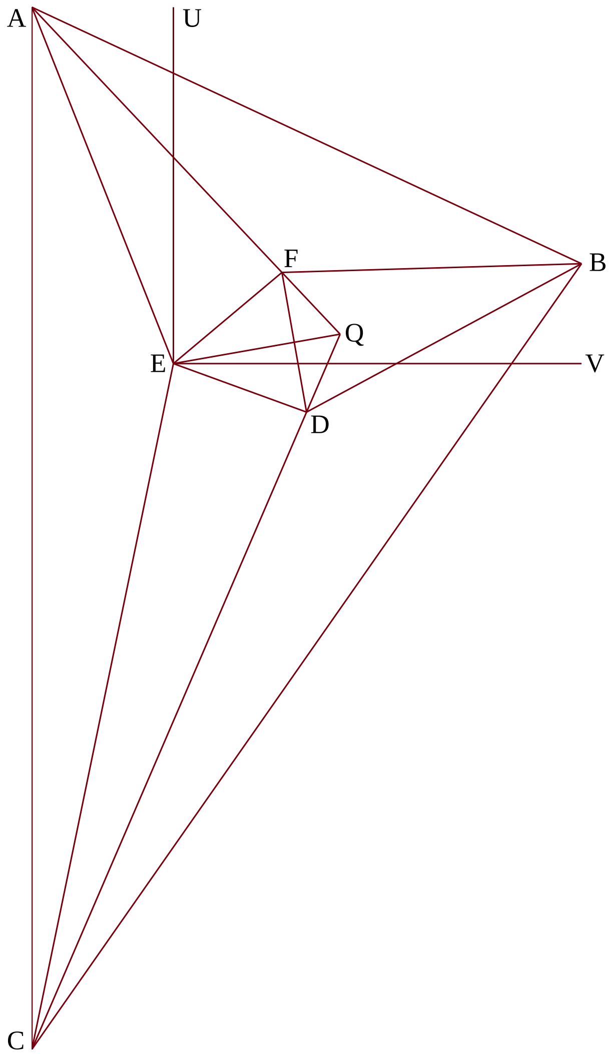

Fig. 4: Showing Morley's Triangle in relation to ΔABC

Since E is the in-centre of ΔAQC, it follows that EQ bisects $\angle CQA$ and hence $\angle DEF$, so that $\angle QEF = \dfrac{\pi}{6}$. It is well-known that $\angle AEF = \dfrac{C+\pi}{3}$, so that $\angle AEQ = \dfrac{C}{3} + \dfrac{\pi}{2}$. Hence EQ makes an angle $\dfrac{C-A}{3} + \dfrac{\pi}{2}$ with EU and an angle $\dfrac{A-C}{3} = 2\beta$ with the horizontal line EV. Thus EQ is parallel to $NCp1$. Since the two lines bisect the angles of their respective triangles at E and $Cp1$, DF is parallel to and FD is antiparallel to $Cp2Cp3$.

2.2 The quartic equation of the deltoid

The Steiner hypocycloid or deltoid is often referred to as the Steiner quartic. It is of interest to obtain the equation in such a form. This requires elimination of ϕ from (4), which is effected by writing the complex number z of (5), supressing the prime, as

$$z = R(e^{2i\phi} + \frac{1}{2}e^{-i(4\phi-6\beta)}), \tag{8}$$

and then

$$\zeta = e^{-2i\beta}z = R(e^{2i\psi} + \frac{1}{2}e^{-4i\psi}), \quad 2\psi = 2\phi - 2\beta. \tag{9}$$

It follows that

$$\begin{aligned} |\zeta|^2 &= |z|^2 = R^2(\frac{5}{4} + \cos 6\psi), \\ \mathrm{Re}\zeta^3 &= R^3(\frac{7}{4}\cos 6\psi + \frac{1}{2}\cos^2 6\psi + \frac{11}{8}), \end{aligned} \tag{10}$$

and hence that

$$16|z\bar{z}|^2 + 27R^2|z\bar{z}| - 27R^4 = 64R\,\mathrm{Re}(e^{-6i\beta}z^3). \tag{11}$$

Eq. (11) does allow the equation to be presented in Cartesian co-ordinates with respect to N as origin. However it is more useful to use polar co-ordinates

$$z = x' + iy' = re^{i\theta}, \tag{12}$$

at N, so that the deltoid appears in the quartic form

$$16r^4 + 72r^2R^2 - 27R^4 = 64r^3R\cos(3(\theta - 2\beta)). \tag{13}$$

The properties of the deltoid that follow from (13) of course agree with those noted above.

When $\theta = 2\beta$, (13) reduces to

$$(2r + R)(2r - 3R)^3 = 0, \tag{14}$$

so that the double tangent to the cusp meets the deltoid again at a point distant $2R$ from the cusp. See Section 7.

An alternative way to identify the cuspidal directions by geometrical consideration setting out from ΔABC and the trisectors of its angles was described long ago in [7], and by a slightly different but pleasing method by [21]. The geometric approach reflects common ground between the Morley context and that of the deltoid. The treatment of [21] allows it to be proved easily by geometrical methods that the sides of the cuspidal triangle are parallel to those of the Morley triangle.

3 Identification of points of contact with the deltoid

Already in [7] a description of the locations of the points (v) and (vi) of tangency to the deltoid can be found. For the case of the side CA of ΔABC and the altitude BE, refer to Fig. 5.

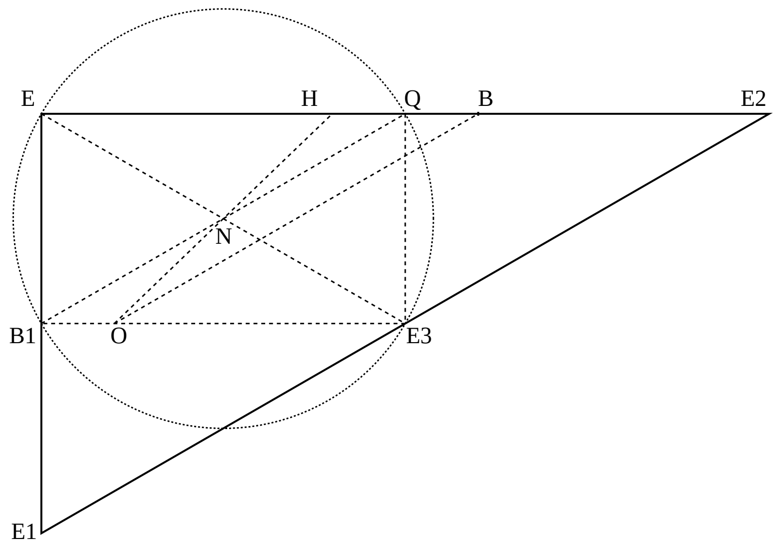

Fig. 5: The points $E1$ and $E2$ and related geometry

The figure shows that
(a) the point $E1$ is the image in the point $B1$ of CA of the foot E of the altitude EB, and
(b) the point $E2$ is the image in the midpoint Q of EH of the foot E of the altitude.

Item (a) can be found in [7] as stated here, but item (b) replaces a less appealing but equivalent one there. Item (a) uses the two points commom to CA and the nine-point circle, to identify $E1$, and item (b) does the same for the altitude BE and $E2$. This underlines the central role of the nine-point circle that is emerging in the overall picture, but there is more about that in Sections 5 and 6. Analogous statements, and figures like Fig. 5, can of course be given for the pair BC and AD, and for the pair AB and CF. For the method of proof of the tangency statements see Section 4. In Section 5, a miscellany of geometrical results made evident by Fig. 5 is presented.

4 Proofs for data for points of tangency to the deltoid

This is done in the same way for the seven lines $BC, CA, AB, AD, BE, CF, E1E2$, and is illustrated for AB. The work is best done using equations referred to origin N and axes there parallel to the van Horn choice defined above. The equation (3) of the Simson line $S(X)$ now reads as

$$y' \cos(3\beta - \phi) - x' \sin(3\beta - \phi) = \frac{R}{2} \sin(3\phi - 3\beta), \tag{15}$$

which is also the equation of the tangent to the deltoid at

$$(x'(\phi), y'(\phi)) = (x(\phi) - x_N, y(\phi) - y_N). \tag{16}$$

The equation of AB required for comparison is

$$y' \sin(\alpha + 3\beta) + x' \cos(\alpha + 3\beta) = -\frac{R}{2} \cos(3\alpha + 3\beta), \tag{17}$$

which is the Simson line $S(X)$ for $2\phi = \pi - 2\alpha$. The co-ordinates of the point $F1$ where AB touches the deltoid are then obtained by evaluating the parametric equation (4) of the deltoid for this value of ϕ. After translating back to the van Horn system at O, the expressions shown in (60) for $F1$ arise.

For the lines listed above, the values of 2ϕ used are, in order,

$$2\alpha + \pi, \quad 6\beta + \pi, \quad \pi - 2\alpha, \quad 2\alpha, \quad 6\beta, -2\alpha, -6\beta. \tag{18}$$

These values show that the seven lines are the Simson lines of the points diametrically opposite $A1, B1, C1$, of the points A, B, C, and of the image of B in the perpendicular bisector of CA.

5 Geometrical facts evident from Fig.5

The points $B1$ and Q are midpoints of CA and BH by definition, and of $EE1$ and $EE2$ by the image statements (a) and (b) in Section 3. Clearly $E1E2$ is parallel to $B1O$ and twice its length. The midpoint $E3$ of $E1E2$ is a member of the important triple (vii) of points. Substitution of the result of (64) for $E3$ into the equation of the nine-point circle is one way to confirm that $E3$ lies on it. Also $EE3$ subtends a right angle at each of $B1$ and Q, and so, like $B1Q$, is a diameter. It follows that

$$B1N = NQ = EN = NE3 = R/2, \tag{19}$$

and

$$B1Q = EE3 = E3E2 = EE3 = R \tag{20}$$

The well-known property $ON = NH$ of the Euler line fit nicely into the picture. Further

$$EE1 = 2R|\sin(A - C)|, \quad EE2 = 2R|\cos(A - C)|. \tag{21}$$

Diagrams similar to Fig. 5 can be drawn for the situations of BC and AD, and of AB and CF. Identical features are seen. In particular, $D3$ and $F3$ lie on the nine-point circle. Also AO, BO, CO are parallel to $A1P, B1Q, C1R$ and to $D1D2, E1E2, F1F2$.

The lines $D1D2, E1E2, F1F2$ have various interesting properties, discussed in Sections 7 and 8. This includes the fact that they touch the deltoid at the triple

(ix) $D4, E4, F4$ of points.

6 From the nine-point circle to the dual triangle

Section 5 establishes that there are now twelve points of geometrical significance on the nine-point circle. These consists of the three triples (ii), (iii), (iv), namely

$$\{A1, B1, C1\}, \quad \{D, E, F\}, \quad \{P, Q, R\}, \tag{22}$$

plus the additional triple (vii) identified in Section 5

$$\{D3, E3, F3\}. \tag{23}$$

By considering the joins of corresponding points of distinct pairs of triples a pattern of concurrency results is realised:

(a) $A1P$, $B1Q$, $C1F$, concurrent at N,
(b) $DD3$, $EE3$, $FF3$, concurrent at N,
(c) PD, QE, RF, concurrent at H,
(d) $A1D3$, $B1E3$, $C1F3$, concurrent at O.

Since the pairs of points $A1P, B1Q, C1R$ lie at opposite ends of diameters of the nine-point circle, (a) is correct. Asking what points lie at the opposite ends of such diameters from the triple D, E, F is already enough to motivate a search for a fourth triple of points to examine. However the points $D3, E3, F3$ already found play the role in question, making (b) clear too. Also (c) is correct, since PD is part of the altitude AD of ΔABC, etc. To prove (d), it is sufficient to show that O lies on each of three lines in question. It is the addition of the points (23) that brings out the symmetry of the system, and the involvement of the Euler line in it.

Of course, it is known also that (d) is true, since $A1O, B1O, C1O$ are the perpendicular bisectors of the sides of ΔABC, but (d) as written extends $A1O, B1O, C1O$ in both directions, and thereby brings the situation of O fully into parity with that enjoyed by H from the outset.

Cases (a) to (d) consider only four out of the six possible cases. The remaining two define triangles. The lines $A1D, B1E, C1F$ are segments of the lines

(e) BC, CA, AB and just lead back to ΔABC.

But the lines

(f) $PD3, QE3, RF3$, when suitably extended in each direction so as to intersect at points $A2, B2, C2$, define a new triangle, triangle $\Delta A2B2C2$, which has been referred to as the Eulerian image triangle of of ΔABC. Clearly

$$PD3||BC, \quad QE3||CA, \quad RF3||AB. \tag{24}$$

Indeed $\Delta A2B2C2$ and ΔABC are congruent, and the lines $AA2, BB2, CC2$ concur at N.

The arrival of $\Delta A2B2C2$ on the scene completes the caste of players in the larger geometrical picture promised in the introduction. It embodies two-fold symmetry Σ under a rotation of 180 degrees about N, wherein ΔABC and $\Delta A2B2C2$ are interchanged, the Euler line rotates so that O and H exchange positions, as do the pair P and $A1$ and the pair D and $D3$, etc. Thus

(1) P, Q, R now play the role of the midpoints of the sides $B2C2, C2A2, A2B2$ of $\Delta A2B2C2$,

(2) $A1, B1, C1$ are the midpoints of the joins $OA2, OB2, OC2$ of the orthocentre O of $\Delta A2B2C2$ to the vertices of $\Delta A2B2C2$.

(3) $D3, E3, F3$ are the feet on $B2C2, C2A2, A2B2$ of the altitudes $A2D3, B2E3, C2F3$ of $\Delta A2B2C2$.

(4) By considering the deltoid of $\Delta A2B2C2$, the fact that D, E, F for it play the same role as $D3, E3, F3$ do for ΔABC can be shown.

There are many issues that might well be investigated at this point. These include some further triples of points.

(x) The image points $D5, E5, F5$ of $D3, E3, F3$ in P, Q, R on $B2C2, C2A2, A2B2$. These are the counterparts for $\Delta A2B2C2$ of $D1, E1, F1$ for ΔABC.

(xi) The image points $D6, E6, F6$ of $D3, E3, F3$ in $A1, B1, C1$ on $A2D3, B2E3, C2F3$. These are the counterparts for $\Delta A2B2C2$ of $D2, E2, F2$ for ΔABC.

Clearly the lines $D1D5, E1E5, F1F5$ are concurrent at N, as are the lines $D2D6, E2E6, F2F6$. The concurrency of the sets of three lines $D1D2, E1E2, F1F2$ and of $D5D6, E5E6, F5F6$, possibly introducing new triangle centres, is discussed in Section 9.

None of the points (ix), (x), (xi) are included in any of the Figures. The co-ordinates only of $E4, E5, E6$ are given in Appendix A. See (65).

In preparation for the developments of Section 9, the doubly Eulerian nature of the Euler line $LOGNH$ of ΔABC is seen in the length ratios $LO : OG : GN : NH = 6 : 2 : 1 : 3$, amd $HG : GO : OL = 4 : 2 : 6 = 2 : 1 : 3$. The map Σ then gives rise to the Euler line

$$L2, O2 = H, G2, N2 = N, H2 = O \tag{25}$$

of $\Delta A2B2C2$. On the common Euler line of the two triangles the lengths of the successive intervals between the points

$$L, O = H2, G, N = N2, G2, H = O2, L2 \tag{26}$$

are in the ratios $6 : 2 : 1 : 1 : 2 : 6$, symmetrical about $N = N2$. The centroid and de Longchamps points $G2$ and $L2$ of $\Delta A2B2C2$ are not new triangle centres, but give rise to an additional role for $X(381)$ and $X(382)$ [14].

7 The lengths of tangents to the deltoid

The lines $D1D2, E1E2, F1F2$ ilustrate a general property [3], item 12 on page 79, of tangents to the deltoid: the tangent to the deltoid terminated at the points where it meets the deltoid again, is of constant length, here $2R$, with a midpoint lying on the inscribed circle. The inscribed circle referred to is that of the deltoid, that is, the nine-point circle of ΔABC. In the case of the three lines $D1D2, E1E2, F1F2$, it remains to show that they are tangent to the deltoid, and to determine the co-ordinates of the points $D4, E4, F4$ of contact. In the cases of the lines BC, CA, AB and AD, BE, CF, their tangency is already known, and it is necessary to find their 'termination points' and verify that the distance between them is $2R$. These matters are attended to in the cases of $E1E2, CA, EB$ in turn.

(a) The case of $E1E2$.

The line $E1E2$ has the equation

$$y + R\sin 6\beta = \tan 6\beta(x - R\cos 2\alpha). \tag{27}$$

As in Section 4, comparing this with (15) for the Simson line $S(X)$, it is seen that $E1E2$ is the Simson line for $2\phi = -6\beta$, so that it touches the deltoid at a point $E4$ with co-ordinates

$$R(\cos 2\alpha + \cos 6\beta + \cos 6\beta\cos 12\beta, \quad \sin 6\beta\cos 12\beta), \tag{28}$$

lying on $E3E2$ at a distance $R\cos 12\beta = R\cos 2(A - C)$ from $E3$. Treatment of $D1D2$ and $F1F2$ works through similarly.

(b) The case of CA.

The line has equation $x + 2R\cos 2\alpha = 0$. It intersects the deltoid, where

$$0 = \cos 2\phi\left[1 + \cos(2\phi - 6\beta)\right]. \tag{29}$$

The point of the deltoid such that $2\phi = \pi + 6\beta$ is the point $R(\cos 2\alpha, \quad -\sin 6\beta)$, that is, the known point $E1$ of tangency with the deltoid. When $\cos 2\phi = 0$, the points of the deltoid are

$$R(\cos 2\alpha, \quad \pm 1) = (x_{B1}, y_{B1} \pm R), \tag{30}$$

lying at a distance $2R$ apart. Also $B1$ is the mid-point of the join of the points (30),and the circle with center $B1$ and radius R passes through Q and $E3$ as well as the points (30).

(c) The case of BE.

The line BE or $EBE2$ has the equation $y = R\sin 6\beta$. It intersects the deltoid where

$$0 = \sin 2\phi\left[1 - \cos(2\phi - 6\beta)\right]. \tag{31}$$

When $2\phi = 6\beta$, the known point $E2$ is found. When $\sin 2\phi = 0$ the points $(x_Q \pm R, \quad y_Q)$, are found, as required, with Q, also as required, on the nine-point circle. The circle with centre Q and a diameter defined by the points (31) passes through $B1$. The line joining the points of intersection of the two circles mentioned in (b) and (c) passes through N.

Results for the remaining cases of each of (b) and (c) can be written down immediately and proved to be correct.

8 Points of concurrency and triangle centres

Many cases of sets of three lines concurrent at a point identifiable as a triangle centre [14] are met with the course of the present study. These include:

(a) the cases discussed in Section 6 involving the familiar triangle centres O, N, H,

(b) lines $AA2, BB2, CC2$, joining the vertices of ΔABC to the corresponding vertices of the dual triangle $\Delta A2B2C2$, also noted in Section 6,

(c) the lines $AD1, BE1, CF1$ concurrent at the triangle centre $X(69)$ of [14], denoted by letter Z here,

(d) lines $D1D2, E1E2, F1F2$ concurrent at a point denoted by the letter K here, not so far located in [14],

(e) lines $D5D6, E5E6, F5F6$ concurrent at a point denoted by the letter J here, which is the image under the symmetry Σ of K.

Although results like those in (a) and (b) are often obvious or easily proved by elementary means, the subject is in general most naturally approached in terms of trilinear coordinates. For information about trilinear co-ordinate, see [12, 14] or go on-line to [22, 23]. A brief discussion of the topic is given in Section 8.1 followed by treatment of Z, K, J. For appreciation of the underlying geometry see [24], chapter3.

All the cases found under (a) and (b) can be treated using trilinears and elementary manipulation of the relevant determinants.

8.1 Use of trilinear co-ordinates

The directed trilinear co-ordinates, or simply the trilinears, of a point W in the plane of ΔABC differ by a common factor, the conversion factor, from the perpendicular distances from the sides BC, CA, AB of ΔABC. If W has trilinears $w_A : w_B : w_C$, then the distances in question, also referred to as the exact trilinear co-ordinates of W, are (ww_A, ww_B, ww_C) where w is given by

$$w(aw_A + bw_B + cw_C) = 2\Delta, \tag{32}$$

where Δ denotes the area of ΔABC. The signs of directed trilinears are chosen so that, for example, w_A is positive if W is on the same side of BC as A, and otherwise negative. In general to check assignments, it is wise to verify that w as calculated from its trilinears using (32) agrees with the value provided by evaluating distances by geometry. Appendix C collects the trilinears of points of interest in Table 1. All the conversions factors w are given there too. To make permutational relations between trilinears of sets of related points manifest, it is best in trilinear work to use the angles A, B, C of ΔABC rather than the van Horn variables α, β.

8.2 The lines $AD1, BE1, CF1$

One method of proof of their concurrency depends on the concept of isotomic conjugate points [5], page 157. It is simplest to adapt the entry explaining the idea in wikipedia [25] to the present case. Thus: if H is not collinear with any pair of the vertices of ΔABC, let AH, BH, CH meet BC, CA, AB in D, E, F, and let $D1, E1, F1$ be the images of D, E, F in the points $A1, B1, C1$ on BC, CA, AB. Then the pairs D and $D1$, E and $E1$, F and $F1$ of points are isotomic conjugates, and the lines $AD1, BE1, CF1$ are concurrent at a point Z which is the isotomic conjugates of H. Further adapting the discussion in the entry in question, as H has trilinear co-ordinates

$$\sec A \ : \ \sec B \ : \ \sec C, \tag{33}$$

then it follows that those of Z are

$$\frac{\cos A}{\sin^2 A} \ : \ \frac{\cos B}{\sin^2 B} \ : \ \frac{\cos C}{\sin^2 C}. \tag{34}$$

To provide an independent proof of the concurrency it is sufficient to show that Z lies on the line $AD1$. This requires the vanishing of the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \sin^2 C\cos B & \sin^2 B\cos C \\ \dfrac{\cos A}{\sin^2 A} & \dfrac{\cos B}{\sin^2 B} & \dfrac{\cos C}{\sin^2 C} \end{vmatrix}, \tag{35}$$

which obviously holds. The trilinears of D are read from the table in Appendix C. The point Z is identified as $X(69)$ of [14], by its trilinears (34).

8.3 The point of concurrency of the lines $D1D2,\ E1E2\ F1F2$

While the required trilinears can be read from Table 1 in Appendix C, some comment on the results for $D2$ is included. The minus signs reflect the fact it lies on the negative side of each of CA and AB. It is of interest to check the results given against the knowledge that a scale factor $k = 2R$ is needed to obtain the actual perpendicular distances, $2R\cos(B-C)$ to BC, for example. Consistency requires that

$$\sin A\ \cos(B-C) - \sin B\ \cos^2 C\ \cos B - \sin C\ \cos^2 B\ \cos C = \frac{\Delta}{2R^2} \tag{36}$$

hold. To treat this, the sine rule is used, allowing proof to be completed with the aid of well-known identities

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C = \frac{2\Delta}{R}. \tag{37}$$

A straightforward way to demonstrate the concurrency of the lines $D1D2,\ E1E2\ F1F2$ uses the equations of the lines given in (68) to find the co-ordinates of the point K of intersection of $D1D2$ and $E1E2$, and then to prove that this K lies on $F1F2$. The results obtained are

$$x_K = R(\cos 2\alpha + \cos 6\beta + \cos 4\alpha\cos 6\beta), \quad y_K = R(\cos 4\alpha\sin 6\beta). \tag{38}$$

It follows from (38) that K lies at a perpendicular distance $2R\cos^2 B\cos(C-A)$ from CA. This suggests that the trilinear co-ordinates of K are given by

$$\cos^2 A\cos(B-C) : \cos^2 B\cos(C-A) : \cos^2 C\cos(A-B). \tag{39}$$

To check the consistency of the value $w = 2R$ of the conversion factor with (32) requires that the result (54) or

$$\sin 2A\cos A\cos(B-C) + \sin 2B\cos B\cos(C-A) + \sin 2C\cos C\cos(A-B) = \frac{\Delta}{R^2}, \tag{40}$$

hold. The proof is completed with the aid of (37).

Alternatively, guessing or knowing the assignment (39), it is sufficient for a proof of the concurrency of the three lines $D1D2,\ E1E2\ F1F2$ to show that K lies on $D1D2$. This follows, if, using entries from Appendix C,

$$\begin{vmatrix} 0 & \sin^2 C\cos B & \sin^2 B\cos C \\ \cos(B-C) & -\cos^2 C\cos B & \cos^2 B\cos C \\ \cos^2 A\cos(B-C) & \cos^2 B\cos(C-A) & \cos^2 C\cos(A-B) \end{vmatrix} = 0. \tag{41}$$

A sequence of elementary determinantal steps makes this answer just fall out.

Some further results that can be obtained with the aid of (38) include

$$KD1 = 2R\cos^2 A, \quad KE1 = 2R\cos^2 B, \quad KF1 = 2R\cos^2 C, \tag{42}$$

$$KD2 = 2R\sin^2 A, \quad KE2 = 2R\sin^2 B, \quad KF2 = 2R\sin^2 C. \tag{43}$$

8.4 The point J of concurrency of the lines $D5D62,\ E5E6\ F5F6$

As J is the image of the point K under the symmetry Σ, then its co-ordinates can be written as $(2x_N - x_K,\quad 2y_N - y_K)$, and obtained explicitly as in (66). Also, using information from (64), the line $E5E6$ has the equation

$$y = \tan 6\beta(x - \cos 2\alpha + \cos 6\beta), \tag{44}$$

so that it can be checked that J lies on it. Proceeding similarly for the other two lines, it follows that J is the required point of concurrency.

The place of K and J in the scheme of [14] is the subject of the next section.

9 The points K, N, J and the Euler line of ΔDEF

The evolution of ideas in the present paper has brought the point K into evidence as the point where the lines $D1D2, E1E2, F1F2$ meet. It is now shown how to put K in its natural place within [14]. It is known from [14] that $N = X(5), X(51) = S, X(143) = T, X(52) = J$, are circumcentre, centroid, nine-point centre, orthocentre of the orthic [14] triangle ΔDEF of ΔABC, where the labels S, T have been used for the points $X(51)$ and $X(143)$. The status assigned here to N follows the fact that the nine-point circle passes through D, E, F, and that of J and or S can be seen by adding a few lines to Fig. 3. For ΔDEF the circumcentre and orthocentre lie outside the triangle itself in the figures as drawn here.

The points K, N, S, T, J have trilinears for which the second entries can be expressed in the form

$$\cos(C - A)(\cos^2 B, \quad 1, \quad 1 - \cos^2 B, \quad 3 - 4\cos^2 B, \quad 2\cos^2 B - 1). \tag{45}$$

Since all the quantities in (45) are linear combinations of 1 and $\cos^2 B$, it follows by the standard determinantal method that all five points are collinear. By working out the scale factors needed to convert trilinear co-ordinates to distances from the sides of ΔABC, it is found using (45) that the distances of the points K, N, S, T, J, from the y'-axis, that is, the line parallel to CA that passes through N, are the following multiples of

$$-6, 0, 2, 3, 6 \quad \text{of} \quad \frac{1}{12}R(1 - 4\cos^2 B)\cos(C - A), \tag{46}$$

with distances between successive points given by $6 : 2 : 1 : 3$. Hence $KNJTS$ is the doubly Eulerian Euler line of the orthic triangele ΔDEF of ΔABC, and K fits nicely into the picture [14] as the de Longchamps point of ΔDEF.

Under the map Σ, the Euler line of the orthic triangle $D3E3F3$ of $\Delta A2B2C2$ is realised with points $K3 = J, N3 = N, S3, T3, J3 = K$. With $K3, N3 = N, J3$ the orthocentre, nine-point centre and de Longchamps point of $\Delta D3E3F3$, the situation is similar to that described at the end of Section 6. The line

$$K = J3, T3, S3, N = N3, S, T, J = K3 \tag{47}$$

is the common doubly Eulerian Euler line of the two orthic triangles, with distances between successive points in the ratios $6 : 2 : 1 : 1 : 2 : 6$. To verify the truth of this statement, add to (45) the entries for $T3$ and $S3$

$$\cos(C - A)(1 + 4\cos^2 B\ ,\ 1 + 2\cos^2 B), \tag{48}$$

and find the scale factors needed to pass from to exact trilinears. They are, with the entries for $T3$ and $S3$ in second and third place

$$2R, \frac{1}{4}R, \frac{1}{3}R, \frac{1}{2}R, \frac{2}{3}R, \frac{1}{4}R, -R. \tag{49}$$

For comments relating to a search for places within [14] for the points $T3$ and $S3$, go to Section 9.1.

9.1 The common Euler line of triangles DEF and $D3E3F3$

The line (47) is the line (5 51) of the Table of Central Lines in [14]. There are these triangle centres listed as lying upon it, centres denoted [14] by $X(n)$ with n given by

$$5, 51, 52, 143, 343, 973, 1154, 1209, 1568. \tag{50}$$

Are any of the last five related perhaps to either of $S3, T3$? By adding the entry $\cot B \cos(C - A)$ or $2ca \cos B \cos(C - A)$ to (45), and writing $2ca \cos B = (a^2 + b^2 + c^2) - 8R^2 \sin^2 B$, it can be seen that $X(343)$ does lie on (5 51), but is distinct from either one of $S3, T3$. The point $X(1154)$ is the point at infinity on (5 51), as agrees the use of its trilinears to give w times 0 for the right side of (32). The trilinears of the remaining three centres of (50) are not given in sufficiently tractable form to allow any immediate conclusion. However the statement [14] that $X(973)$ is the midpoint of the join of $X(52)$ and $X(1209)$ allows it to be inferred that neither $X(973)$ nor $X(1209)$ can be identified with either one of $S3, T3$. No comment is offered regarding $X(1568)$.

9.2 Identities useful for evaluating scale factors

Note two for each of Sections 6 and 9.

$$\begin{aligned}
a\cos A + b\cos B + c\cos C &= \frac{2\Delta}{R} && (51)\\
a\cos B\cos C + b\cos C\cos A + c\cos A\cos B &= \frac{\Delta}{R} && (52)\\
a\cos(B-C) + b\cos(C-A) + c\cos(A-B) &= \frac{4\Delta}{R} && (53)\\
a\cos^2 A\cos(B-C) + b\cos^2 B\cos(C-A) + c\cos^2 C\cos(A-B) &= \frac{\Delta}{R}. && (54)
\end{aligned}$$

Appendix A: Expressions for the co-ordinates of points of interest

$$\begin{array}{lll}
A & R(\cos 2\alpha, & \sin 2\alpha)\\
B & R(\cos 6\beta, & \sin 6\beta)\\
C & R(\cos 2\alpha, & -\sin 2\alpha)
\end{array} \qquad (55)$$

$$\begin{array}{lll}
A1 & \frac{1}{2}R(\cos 2\alpha + \cos 6\beta, & -\sin 2\alpha + \sin 6\beta)\\
B1 & R(\cos 2\alpha, & 0)\\
C1 & \frac{1}{2}R(\cos 2\alpha + \cos 6\beta, & \sin 2\alpha + \sin 6\beta)
\end{array} \qquad (56)$$

$$\begin{array}{lll}
D & R(\cos 2\alpha + \sin 2\alpha\sin(2\alpha - 6\beta), & \sin 2\alpha\cos(2\alpha - 6\beta))\\
E & R(\cos 2\alpha, & \sin 6\beta)\\
F & R(\cos 2\alpha + \sin 2\alpha\sin(2\alpha + 6\beta), & -\sin 2\alpha\cos(2\alpha + 6\beta))
\end{array} \qquad (57)$$

$$\begin{array}{lll}
P & R(\frac{3}{2}\cos 2\alpha + \frac{1}{2}\cos 6\beta, & \frac{1}{2}\sin 2\alpha + \frac{1}{2}\sin 6\beta)\\
Q & R(\cos 2\alpha + \cos 6\beta, & \sin 6\beta)\\
R & R(\frac{3}{2}\cos 2\alpha + \frac{1}{2}\cos 6\beta, & -\frac{1}{2}\sin 2\alpha + \frac{1}{2}\sin 6\beta)
\end{array} \qquad (58)$$

$$\begin{array}{lll}
O & (0, & 0)\\
H & R(2\cos 2\alpha + \cos 6\beta, & \sin 6\beta)\\
N & R(\cos 2\alpha + \frac{1}{2}\cos 6\beta, & \frac{1}{2}\sin 6\beta)
\end{array} \qquad (59)$$

$$
\begin{array}{cc}
D1 & R(\cos 2\alpha\cos(2\alpha-6\beta), -\sin 2\alpha - \cos 2\alpha\sin(2\alpha-6\beta)) \\
E1 & R(\cos 2\alpha, \quad -\sin 6\beta) \\
F1 & R(\cos 2\alpha\cos(2\alpha+6\beta), \quad \sin 2\alpha + \cos 2\alpha\sin(2\alpha+6\beta))
\end{array} \tag{60}
$$

$$
\begin{array}{cc}
D2 & R(2\cos 2\alpha + \cos 2\alpha\cos(2\alpha-6\beta), \quad \sin 2\alpha - \cos 2\alpha\sin(2\alpha-6\beta) \\
E2 & R(\cos 2\alpha + 2\cos 6\beta, \quad \sin 6\beta) \\
F2 & R(2\cos 2\alpha + \cos 2\alpha\cos(2\alpha+6\beta), \quad -\sin 2\alpha + \cos 2\alpha\sin(2\alpha+6\beta)
\end{array} \tag{61}
$$

$$
\begin{array}{cc}
A2 & R(\cos 2\alpha + \cos 6\beta, \quad -\sin 2\alpha + \sin 6\beta) \\
B2 & R(2\cos 2\alpha, \quad 0) \\
C2 & R(\cos 2\alpha + \cos 6\beta, \quad \sin 2\alpha + \sin 6\beta)
\end{array} \tag{62}
$$

$$
\begin{array}{cc}
D3 & R(\cos 2\alpha + \cos 2\alpha\cos(2\alpha-6\beta), \quad -\cos 2\alpha\sin(2\alpha-6\beta) \\
E3 & R(\cos 2\alpha + \cos 6\beta, \quad 0) \\
F3 & R(\cos 2\alpha + \cos 2\alpha\cos(2\alpha+6\beta), \quad \cos 2\alpha\sin(2\alpha+6\beta)
\end{array} \tag{63}
$$

$$
\begin{array}{cc}
E4 & R(\cos 2\alpha + \cos 6\beta + \cos 6\beta\cos 12\beta, \quad \sin 6\beta\cos 12\beta) \\
E5 & R(\cos 2\alpha + \cos 6\beta, \quad 2\sin 6\beta) \\
E6 & R(\cos 2\alpha - \cos 6\beta, \quad 0)
\end{array} \tag{64}
$$

$$
K \quad R(\cos 2\alpha + \cos 6\beta + \cos 4\alpha\cos 6\beta, \quad \cos 4\alpha\sin 6\beta) \tag{65}
$$

$$
S \quad R(\cos 2\alpha + \cos 6\beta - 2\cos^2 2\alpha\cos 6\beta, \quad 2\sin^2 2\alpha\sin 6\beta) \tag{66}
$$

Appendix B : Equations of lines

The first set of equations involve van Horn choice of axes with origin at O.

$$
\begin{array}{cc}
BC & y + R\sin 2\alpha = \cot(\alpha - 3\beta)(x - R\cos 2\alpha) \\
CA & 0 = x - R\cos 2\alpha \\
AB & y - R\sin 2\alpha = -\cot(\alpha + 3\beta)(x - R\cos 2\alpha)
\end{array} \tag{67}
$$

$$
\begin{array}{cc}
AD & y - R\sin 2\alpha = -\tan(\alpha - 3\beta)(x - R\cos 2\alpha) \\
BE & y - R\sin 6\beta = 0 \\
CF & y + R\sin 2\alpha = \tan(\alpha + 3\beta)(x - R\cos 2\alpha)
\end{array} \tag{68}
$$

$$
\begin{array}{cc}
D1D2 & y + R\sin 2\alpha + R\cos 2\alpha\sin(2\alpha - 6\beta) = \tan 2\alpha(x - R\cos 2\alpha\cos(2\alpha - 6\beta)) \\
E1E2 & y + R\sin 6\beta = \tan 6\beta(x - R\cos 2\alpha) \\
F1F2 & y - R\sin 2\alpha - R\cos 2\alpha\sin(2\alpha + 6\beta) = -\tan 2\alpha(x - R\cos 2\alpha\cos(2\alpha + 6\beta))
\end{array} \tag{69}
$$

$$
\begin{array}{ll}
B2C2 & y = \cot(\alpha - 3\beta)(x - 2R\cos 2\alpha) \\
C2A2 & x = R(\cos 2\alpha + \cos 6\beta) \\
A2B2 & y = -\cot(\alpha + 3\beta)(x - 2R\cos 2\alpha)
\end{array}
\tag{70}
$$

$$
\begin{array}{ll}
A2D3 & y = -\tan(\alpha - 3\beta)x \\
B2E3 & y = 0 \\
C2F3 & y = \tan(\alpha + 3\beta)x
\end{array}
\tag{71}
$$

Next, versions of some of the above relative to origin at N with the same directions for the x- and y-axes.

$$
\begin{array}{ll}
BC & y\sin(\alpha - 3\beta) - x\cos(3\alpha - 3\beta) = \frac{R}{2}\cos(\alpha - 3\beta) \\
CA & 0 = x + \frac{R}{2}\cos 6\beta \\
AB & y\sin(\alpha + 3\beta) + x\cos(\alpha + 3\beta) = -\frac{R}{2}\cos(3\alpha + 3\beta)
\end{array}
\tag{72}
$$

$$
\begin{array}{ll}
AD & y\cos(\alpha - 3\beta) + x\cos(\alpha - 3\beta) = \frac{R}{2}\cos(3\alpha - 3\beta) \\
BE & y - \frac{R}{2}\sin 6\beta = 0 \\
CF & y\cos(\alpha + 3\beta) - x\sin(\alpha + 3\beta) = -\frac{R}{2}\sin(3\alpha + 3\beta)
\end{array}
\tag{73}
$$

$$
E1E2 \qquad y\cos 6\beta - x\sin 6\beta = -R\sin 6\beta\cos 6\beta. \tag{74}
$$

Appendix C: Table of sets of trilinear coordinates

If the trilinear co-ordinates of any point X are written $x : y : z$, then results for points of interest here can be assembled into a Table. The conversion factor w of (32) needed to relate the trilinear co-ordinates of a point to its perpendicular distances from the sides of ΔABC is given too.

Point	x	y	z	w
A	1	0	0	$bc/(2R)$
B	0	1	0	$ca/(2R)$
C	0	0	1	$ab/(2R)$
$A1$	0	$\sin C$	$\sin B$	$a/2$
$B1$	$\sin C$	0	$\sin A$	$b/2$
$C1$	$\sin B$	$\sin A$	0	$c/2$
D	0	$\cos C$	$\cos B$	$bc/(2R)$
E	$\cos C$	0	$\cos A$	$ca/(2R)$
F	$\cos B$	$\cos A$	0	$ab/(2R)$
P	$\cos(B-C)$	$\cos A\cos C$	$\cos A\cos B$	R
Q	$\cos B\cos C$	$\cos(C-A)$	$\cos A\cos B$	R
R	$\cos B\cos C$	$\cos A\cos C$	$\cos(A-B)$	R
$D1$	0	$\sin^2 C\cos B$	$\sin^2 B\cos C$	$2R$
$E1$	$\sin^2 C\cos A$	0	$\sin^2 A\cos C$	$2R$
$F1$	$\sin^2 B\cos A$	$\sin^2 A\cos B$	0	$2R$
$D2$	$\cos(B-C)$	-$\cos^2 C\cos B$	-$\cos^2 B\cos C$	$2R$
$E2$	-$\cos^2 C\cos A$	$\cos(C-A)$	-$\cos^2 A\cos C$	$2R$
$F2$	-$\cos^2 B\cos A$	-$\cos^2 A\cos B$	$\cos(A-B)$	$2R$
$D3$	$\cos(B-C)$	$-\cos B\cos 2C$	$-\cos C\cos 2B$	R
$E3$	$-\cos A\cos 2C$	$\cos(C-A)$	$-\cos C\cos 2A$	R
$F3$	$-\cos A\cos 2B$	$-\cos B\cos 2A$	$\cos(A-B)$	R
$A2$	$-\cos A$	$\cos(C-A)$	$\cos(A-B)$	R
$B2$	$\cos(B-C)$	$-\cos B$	$\cos(A-B)$	R
$C2$	$\cos(B-C)$	$\cos(C-A)$	$-\cos C$	R
N	$\cos(B-C)$	$\cos(C-A)$	$\cos(A-B)$	$R/2$
O	$\cos A$	$\cos B$	$\cos C$	R
H	$\cos B\cos C$	$\cos C\cos A$	$\cos A\cos B$	$2R$
K	$\cos^2 A\cos(B-C)$	$\cos^2 B\cos(C-A)$	$\cos^2 C\cos(A-B)$	$2R$
S	$\cos 2A\cos(B-C)$	$\cos 2B\cos(C-A)$	$\cos 2C\cos(A-B)$	-R

Appendix D: Lengths of lines seen in the Figures

This appendix shows results for ΔABC along with their counterparts for $\Delta A2BC2$. with points (viii),(iv),(vii),(ii),(x),(xi),(iii),O, H for the latter corresponding to the points (i) to (vii) and H, O for the former. Thus the points (ii) and (iv) of the medial and Euler triangles [14] of ΔABC correspond to the vertices of the Euler and medial triangles of $\Delta A2BC2$.

$$BC = B2C2 = a = 2R\sin A, \quad CA = C2A2 = b = 2R\sin B, \quad AB = A2B2 = c = 2R\sin C \tag{75}$$

$$AD = A2D3 = b\sin C = c\sin B = 2R\sin B\sin C = bc/(2R) \tag{76}$$
$$BE = B2E3 = c\sin A = a\sin C = 2R\sin C\sin A = ca/(2R) \tag{77}$$
$$CF = C2F3 = a\sin B = b\sin A = 2R\sin A\sin B = ab/(2R) \tag{78}$$

$$AH = OA2 = 2R\cos A, \quad BH = OB2 = 2R\cos B, \quad CH = OC2 = 2R\cos C \tag{79}$$

$$AP = PH = A2A1 = A1O = R\cos A \tag{80}$$
$$BQ = QH = B2B1 = B1O = R\cos B \tag{81}$$
$$CR = RH = C2C1 = C1O = R\cos C \tag{82}$$

$$DH = OD3 = 2R\cos B\cos C \tag{83}$$
$$EH = OE3 = 2R\cos C\cos A \tag{84}$$
$$FH = OF3 = 2R\cos A\cos B \tag{85}$$

$$DP = D3A1 = PD2 = A1D6 = R\cos(B-C) \tag{86}$$
$$EQ = E3B1 = QE2 = B1E6 = R\cos(C-A) \tag{87}$$
$$FR = F3C1 = RF2 = C1F6 = R\cos(A-B) \tag{88}$$

$$EF = E3F3 = R\sin 2A, \quad FD = F3D3 = R\sin 2B, \quad DE = D3E3 = R\sin 2C \tag{89}$$
$$QR = B1C1 = R\sin A, \quad RP = C1A1 = R\sin B, \quad PQ = A1B1 = R\sin C \tag{90}$$
$$A1P = B1Q = C1R = R, \quad DD3 = EE3 = FF3 = R \tag{91}$$

$$BD = D1C = B2D3 = D3C2 = c\ cosB, \quad CD = C2D3 = b\cos C \tag{92}$$
$$AE = E1C = A2E3 = E5C2 = c\cos A, \quad EC = C2E3 = a\cos C \tag{93}$$
$$BF = F1A = B2F3 = F5A2 = a\cos B, \quad FA = A2F3 = b\cos A \tag{94}$$

$$DA1 = A1D1 = D3P = PD5 = R|\sin(B-C)| = \frac{1}{2}D1D = \frac{1}{2}D3D5 \tag{95}$$
$$EB1 = B1E1 = E3Q = QE5 = R|\sin(C-A)| = \frac{1}{2}E1E = \frac{1}{2}E3E5 \tag{96}$$
$$FC1 = C1F1 = F3R = RF5 = R|\sin(A-B)| = \frac{1}{2}F1F = \frac{1}{2}F3F5 \tag{97}$$

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