On the relative distances between triangle centres lying on the central lines of an arbitrary plane triangle

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Abstract

The paper studies triangle centres and central lines of a suitable reference triangle. It derives formulas for the ratios of the distances between triangle centres along various central lines, including lines on which the Apollonius point, the centre of the Apollonius circle and the midpoint of the join of the two Brocard points lie. It identifies Eulerian lines, some already known, and some new here, these being lines involving the ratios 3 : 1 : 2 like those for $HN : NG : GC'$ on the Euler line itself, where H, N, G, C' denote the orthocentre, the centre of the nine-point circle, the centroid and the circumcentre. Accurate diagrams are presented to allow a unified view to be taken of the triangle centres and of central lines considered in the paper. They motivate consideration of sets of Eulerian lines, all of which pass through the centroid G, and thereby gives rise to a number of theorems of a vectorial type in the geometry of the family of triangle centres.

Keywords: triangle centre, central line, Eulerian line, trilinear coordinate, position vector

AMS Classification: 51N05, 51N20, 51N99

1 Introduction

This paper is concerned with certain central lines, including Eulerian lines, in the plane of an arbitrary acute-angled reference triangle ΔABC, that is not isosceles. A central line, for example the Euler line [1], page 165, [2], page 28, is a line on which there lie three or more triangle centres of ΔABC. The centres on the Euler line are, historically, the orthocentre H, the nine-point centre N, the centroid, G and the circumcentre C', and, post-Euler, the de Longchamps point, L, [3, 4]. Well-known facts about them motivates the

Definition. A set of four triangle centres lying on the same line is said to be Eulerian, as is the line on which they lie, if the distances between the centres taken consecutively along the line are in the ratio 3 : 1 : 2.

The sets $HNGC'$ and $LC'GH$ are both Eulerian, so that the line $HNGC'L$ is a doubly Eulerian line. It appears that the centroid G is always the third point of an Eulerian line and the middle point of a doubly Eulerian line.

This paper obtains, for all the triangle centres P in Tables 1 and 2, explicit formulas for their position vectors $\mathbf{OP}$ relative to an arbitrary origin in the plane of ΔABC. These formulas allow a systematic approach to proofs needed for a wide range of important problems in the geometry of triangle centres and central lines.

A major purpose of this article is the derivation of explicit formulas for the ratios of the lengths of the intervals between triangle centres, taken consecutively along the central lines on which they lie. Proofs of such formulas, summarised below in this introductory section, are found in Section 5. Another purpose is to achieve an overall understanding of a bigger geometric picture involving triangle centres seen as a family. This includes the identification of Eulerian lines. Such lines are notable in their own right, but also because of the many geometrical insights which they provide.

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Results in this area, also summarised later in this introduction, are found in Section 4. Finally knowledge of the position vectors of triangle centres enables the computation of their Cartesian coordinates, relative to the chosen origin O, and suitable axes, and hence the preparation of the geometrically accurate diagrams Figures 1 and 2. These, it is hoped, will aid understanding of the geometry – collinearities, concurrences, parallelisms, etc. – of the general triangle centre scene. Some comment on the Figures is contained in Section 6.

A comprehensive documentation of triangle centres is available [5,6]. The numbering of triangle centres used there is followed, alongside the use of letters, many in widespread use, so that, for example, for the centroid $G = X(2)$. Amongst the properties of centres and central lines contained in [5,6], there is an essentially exhaustive compilation of the collinearities of three or more centres, including the valuable Table of Central Lines [6], which is a numbered listing of such lines. These are referred to here by the number used in the Table. For example, the notations $K(2)$ and $K(3)$ are used according to

$$HNGC'L = K(2), \quad KB'C'Q = K(3). \tag{1}$$

Tables 1 and 2 contain a listing of the triangle centres that interest here is focussed on, together with some other points which acquire interest in view of their role in the geometry of triangle centres and central lines. In fact it is convenient to assemble in these tables a large body of information for reference throughout the sequel. The ten triangle centres

$$I, G, C', H, N, G', N', M, S, L \tag{2}$$

which feature in Table 1 are ones for which the data presented is considered too well-known to require proving here. Moreover, the centres in Table 1 constitute a family of centres for which any linear vector statement involving them admits truly simple proof.

For centres listed in Table 2, expressions and proofs are generally more complicated – note the denominators of the entries in the last column. Full justification of the data presented there is given below, because many of the results believed to be new here relate to the Brocard midpoint B', the Apollonius point A', and the centre Q of the Apollonius circle.

For background on all the centres (2) of Table 1, except L, for which see [3,4], and also for K, F and $B' = X(39)$, see [1], where their role in the geometry of a plane triangle is discussed extensively. For the Apollonius point $A' = X(181)$, see [7], for the Apollonius centre $Q = X(970)$ see [7,8] and for the isogonal conjugate $M' = X(57)$ of the mittenpunkt M, see [9,10].

Many of the key results of this paper are proved in Section 5. In Section 5.1 on $K(3) = KB'C'Q$, the ratio $KB' : B'C'$ is given by (89) while the result (92) for $KC' : C'Q$ confirms a result given in [7,8].

In Section 5.2 on $K(35) = NSQ$, the ratio $NS : SQ$ is given by (97).

In Section 5.3 on $K(600) = QA'I$, the result (100) relates the vectors $\mathbf{IA'}$ and $\mathbf{QA'}$, with discussion of the order of the points along the line.

Section 5.4 contains the proof that $K(98) = EB'GD$ is Eulerian, and mentions a geometrical consequence.

In Section 5.5 on $K(16) = MGJG'M'$, it is noted that the introduction of the point J makes the set of points $G'JGM$ Eulerian, and the ratio $MG : GM'$ is given as (108).

In Section 5.6 on $K(11) = C'IM'$, the ratio $C'I : IM'$ is given as (112).

The ratio formulas listed so far are believed to be new, except as noted. The remaining sections 5.7 – 5.10 deal with lines $K(4) = MIK$, $K(26) = UMSH$, $K(9) = G'IL$, and $K(13) = NIF$, where at most some additions to existing knowledge may be found. For example, the order of the centres, including the two Soddy centres [12–14], along $K(9)$ is treated in detail, augmenting to some extent the information available in [12–14].

The organisation of the rest of the paper is as follows. Section 2.1 attends to matters of notation and gives some associated definitions and trigonometric identities. Section 2.2 proves as Theorem 1 an identity between the quantities Γ_A and Λ_A, defined in section 2.1, which feature crucially in Table 1, and in several other places. Section 3 explains the key role in the sequel of the entries P_A of Tables 1 and 2 showing that the simple basis $\mathcal{B}_A = \{1, a, a^2\}$ suffices for all of them.

$X(P)$	P	Name	TCF	P_A	w(P)
1	I	incentre	1	$\dfrac{a}{2s}$	r
2	G	centroid	$\dfrac{1}{a}$	$\frac{1}{3}$	$\dfrac{2\Delta}{3}$
3	C'	circumcentre	α	$\frac{1}{2}\Gamma_A$	R
4	H	orthocentre	$\beta\gamma$	$1-\Gamma_A$	$2R$
5	N	nine-point centre	$\alpha+2\beta\gamma$	$\frac{1}{2}-\frac{1}{4}\Gamma_A$	$\dfrac{R}{2}$
7	G'	Gergonne point	$\dfrac{b'c'}{a}$	Λ_A	$\dfrac{2\Delta}{q-s^2}$
8	N'	Nagel point	$\dfrac{a'}{a}$	$1-\dfrac{a}{s}$	$2r$
9	M	mittenpunkt	a'	$\frac{1}{2}-\frac{1}{2}\Lambda_A$	$\dfrac{\Delta}{q-s^2}$
10	S	Spieker centre	$\dfrac{2s-a}{a}$	$\frac{1}{2}-\dfrac{a}{4s}$	$\dfrac{r}{2}$
20	L	de Longchamps point	$\alpha-\beta\gamma$	$2\Gamma_A-1$	$2R$
	J			$\frac{1}{4}+\frac{1}{4}\Lambda_A$	
	T			$\Gamma_A-\frac{1}{2}+\frac{1}{2}\Lambda_A$	
946	W			$\dfrac{a}{4s}+\frac{1}{2}-\frac{1}{2}\Gamma_A$	
40	U	Bevan point	$aa'-r^2-2rR$	$\Gamma_A-\dfrac{a}{2s}$	
962	V			$1+\dfrac{a}{s}-2\Gamma_A$	

Table 1: Triangle Centres

Section 3 provides detailed derivations of the entries P_A for the centres of Table 2, and the role of the scaling factor $w(P)$ in relating general TLC and the TCF to exact TLC is described there.

Section 4 contains

(i) the definitions of the points

$$J, T, W, U, V, \tag{3}$$

(ii) four theorems which provide part of the motivation for including the centres (3) in the present study,

(iii) the identification of a number of Eulerian lines, and discussion of the geometrical results that stem from the interrelationships amongst them,

(iv) comment on sets of central lines, which are parallel, and which therefore share the same point at infinity,

(iv) the identification of the points W, U, V of Table 1, defined geometrically in Section 4, with triangle centres already present in the listing in [6]. This reveals another reason for regarding the points W, U, V as being interesting: there are only four centres G, U, W, V listed [6] on the line $K(98)$ and they constitute an Eulerian set $VWGU$.

Sections 5 and 6 have already been mentioned. An appendix provides a selection of expressions for the actual distances between triangle centres. These results, when supplemented by equalities found throughout the paper, such as those in Section 4.2, suffices to yield a result for the distance between a large number of pairs of triangle centres.

$X(P)$	P	Name	TCF	P_A	$w(P)$
6	K	symmedian point	a	$\dfrac{a^2}{2(2s^2-q)}$	$\dfrac{\Delta}{2s^2-q}$
11	F	Feuerbach point	$1-\cos(B-C)$	$\dfrac{Ra(1-\cos(B-C))}{2s(R-2r)}$	$\dfrac{rR}{R-2r}$
39	B'	Brocard midpoint	$a(b^2+c^2)$	$\dfrac{1}{2}-\dfrac{b^2c^2}{2(q^2-4st)}$	$\dfrac{\Delta}{q^2-4st}$
57	M'	isogonal conjugate of M	$b'c'$	$\dfrac{b'c'-r^2}{2r(2R-r)}$	$\dfrac{1}{2R-r}$
181	A'	Apollonius point	$a(b+c)^2b'c'$	Eq. (47)	$\dfrac{1}{2Rs(s^2-3r^2)}$
970	Q	Apollonius centre	Eq. (42)	Eq. (45)	$-\dfrac{1}{16\Delta t}$
76	D	third Brocard point	b^3c^3	$\dfrac{b^2c^2}{q^2-4st}$	$\dfrac{1}{2R(q^2-4st)}$
194	E		$b^3c^3-at(b^2+c^2)$	$1-\dfrac{2b^2c^2}{q^2-4st}$	$-\dfrac{1}{2R(q^2-4st)}$
176	Σ	inner Soddy centre	$1+\dfrac{\Delta}{aa'}$	$\dfrac{a+b'c'/r}{2s+r+4R}$	$\dfrac{2\Delta}{2s+r+4R}$
175	Σ'	outer Soddy centre	$1-\dfrac{\Delta}{aa'}$	$\dfrac{a-b'c'/r}{2s-(r+4R)}$	$\dfrac{2\Delta}{2s-r-4R}$

Table 2: More triangle centres

2 Preliminaries

2.1 Notation and identities

The letters s, r, R and $\Delta = rs$, denote the semiperimeter, the radius of the incircle, the radius of the circumcircle and the area of ΔABC. As a basis for expressions invariant under permutation of a, b, c, one choice uses

$$s = (a+b+c)/2, \quad q = bc+ca+ab, \quad t = abc. \tag{4}$$

Another choice, often preferred here for presenting results in their final form, uses s, r, R. The connection between the two sets of invariants involves the identities

$$\begin{aligned} q - s^2 &= r(r+4R) && (5)\\ t &= 4R\Delta = 4Rrs \\ \Delta^2 = r^2s^2 &= qs^2 - s^4 - st, && (6) \end{aligned}$$

and it is worthwhile to employ the abbreviation

$$u = s^2 + r^2 + 2rR. \tag{7}$$

Various results like

$$a^2+b^2+c^2 = 2(2s^2-q) \tag{8}$$
$$b^2c^2+c^2a^2+a^2b^2 = q^2-4st \tag{9}$$
$$a^4+b^4+c^4 = 2(q^2-4st)-16\Delta^2, \tag{10}$$

are needed in the sequel.

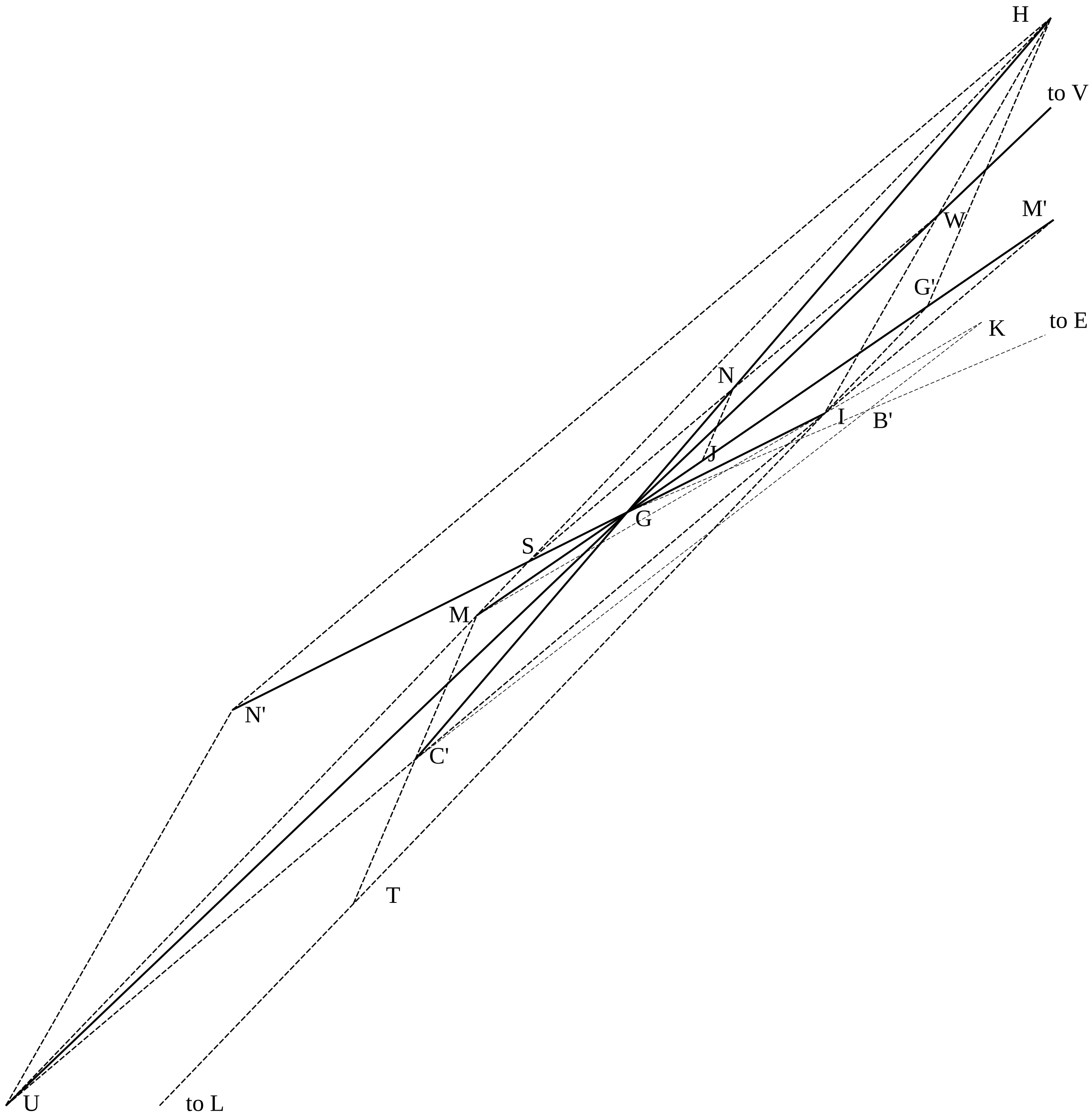

Fig. 1: Triangle centres and central lines

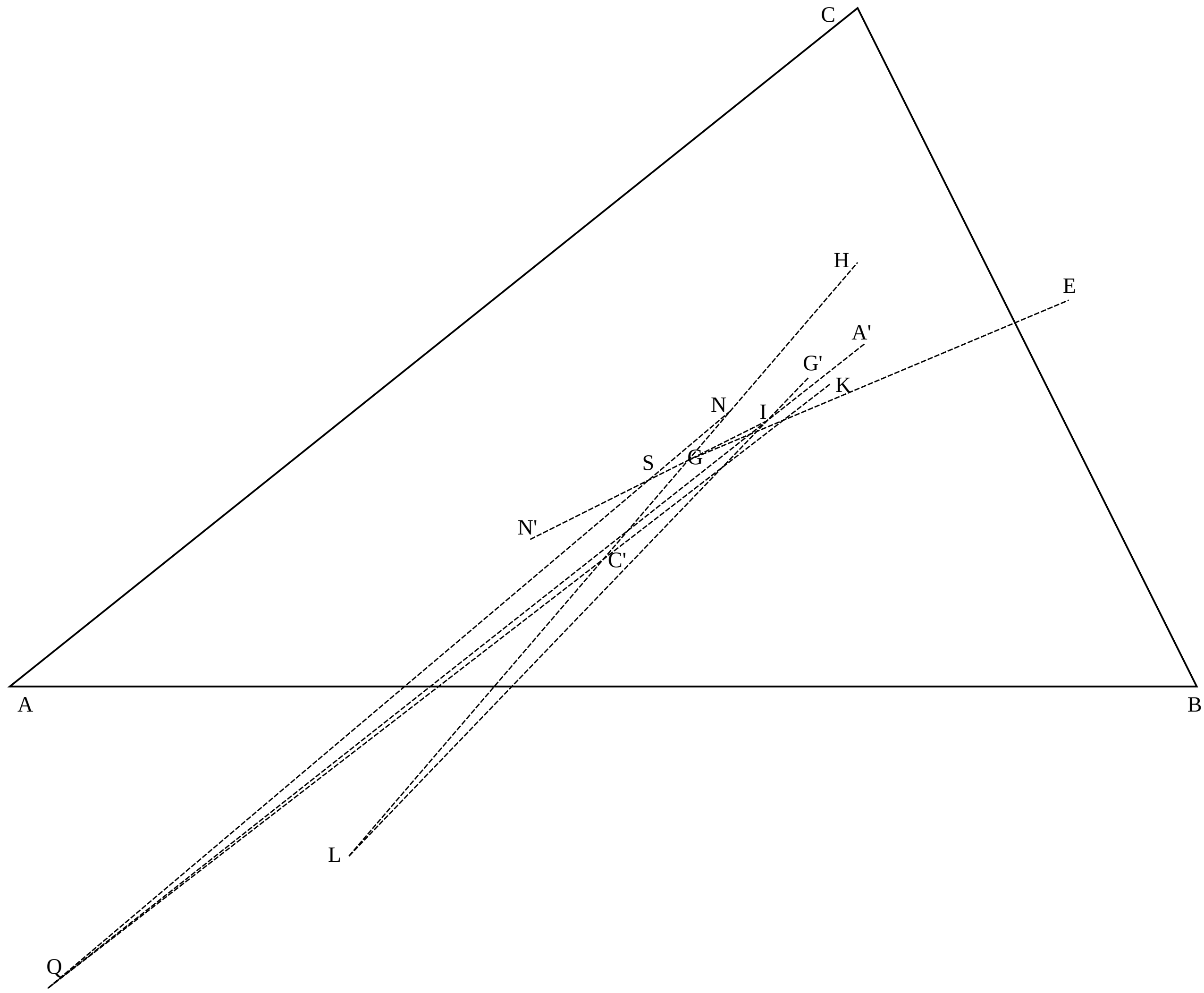

Fig.2: The reference triangle and selected centres and central lines

The Brocard midpoint lies where GE meets $KC'Q$

Define also

$$a' = s - a, \quad b' = s - b, \quad c' = s - c' \tag{11}$$

so that Heron's formula appears as

$$\Delta^2 = sa'b'c'. \tag{12}$$

Further, in Table 1, the notations

$$\alpha = \cos A, \quad \beta = \cos B, \quad \gamma = \cos C \tag{13}$$

are used, along with the non-standard abbreviations

$$\Gamma_A = \frac{\cos A}{\sin B \sin C} = \frac{R}{\Delta} a \cos A, \quad \Lambda_A = \frac{b'c'}{q - s^2}. \tag{14}$$

In view of the role of Λ_A in Table 1, the identity

$$q - s^2 = aa' + b'c' = bb' + c'a' = cc' + a'b', \tag{15}$$

is noted.

2.2 An identity relating Γ_A and Λ_A

Theorem 1. The relation between Γ_A and Λ_A is given by the identity

$$r(r+4R)\Lambda_A = b'c' = r(r+2R)\frac{a}{s} - r^2(2\Gamma_A - 1). \tag{16}$$

Proof. Most proofs seem to be somewhat clumsy. One proof requires two main steps. First

$$\begin{aligned}\frac{b'c'}{r} &= \frac{1}{rs^2}(ac\cos^2\frac{B}{2})(ab\cos^2\frac{C}{2}) \\ &= \frac{Ra}{s}(1+\cos B)(1+\cos C).\end{aligned} \tag{17}$$

The second steps uses

$$\begin{aligned}\cos B\cos C &= \cos(B+C) + \sin B\sin C \\ &= -\cos A + \frac{bc}{4R^2},\end{aligned} \tag{18}$$

and

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}, \tag{19}$$

to give

$$\frac{s}{r}b'c' = Ra(1+\cos B)(1+\cos C) = a(r+2R) - \Delta(2\Gamma_A - 1). \tag{20}$$

Since

$$b'c' = (q-s^2)\Lambda_A = r(r+4R)\Lambda_A, \tag{21}$$

follows use of the definition of Λ_A and of (5), equating expressions for $b'c'$ from (20) and (21) allows the proof of (16) to be completed.

The variables Γ_A and Λ_A are undoubtedly useful. It is important therefore to note that use of (15) and (16) yields

$$r(r+4R)\Lambda_A = b'c' = q - s^2 - aa' = r(r+4R) - as + a^2 \tag{22}$$

and

$$2r^2 s\Gamma_A = -sa^2 + ua - t, \tag{23}$$

where u is given by (7).

Note that it follows from (20) that $(1+\beta)(1+\gamma)$ may be used as the TCF for G', the Gergonne point. This useful alternative to the entry in Table 1 is used in [4] to show that G', I, L are collinear.

Note also, in relation to N, that $\alpha + 2\beta\gamma = \cos(B-C)$. This is compatible with the entry $N_A = \frac{1}{2} - \frac{1}{4}\Gamma_A$ in Table 1 because of the identity

$$a\cos(B-C) = -a\cos A + \frac{2\Delta}{R}. \tag{24}$$

3 Position vectors of triangle centres, and the P_A of Tables 1 and 2

Let A, B, C be the vertices of the reference triangle, ΔABC, which is acute angled and not isosceles. If P is a point in the plane of ΔABC, its position vector $\mathbf{OP}$ with respect to a fixed but arbitrary origin in the plane, can be written in the form

$$\mathbf{OP} = P_A\mathbf{OA} + P_B\mathbf{OB} + P_C\mathbf{OC}. \tag{25}$$

For any triangle centre P, the dependence of the coefficients P_A, P_B, P_C in (25), on the choice made of the triangle of reference, may be expressed entirely in terms of the lengths $a = BC, b = CA, c = AB$ of its sides. Moreover [5, 6], this dependence is such that

$$P_B(a,b,c) = P_A(b,c,a) \quad , \qquad P_C(a,b,c) = P_A(c,a,b), \tag{26}$$
$$P_A(a,b,c) \quad = \quad P_A(a,c,b). \tag{27}$$

For example, the incentre I has the position vector

$$\mathbf{OI} = (a\mathbf{OA} + b\mathbf{OB} + c\mathbf{OC})/(2s). \tag{28}$$

To prove for such points, for example, that $\mathbf{OP_1} + \mathbf{OP_2} = \mathbf{OP_3}$, it is clearly sufficient to show that

$$(P_1)_A + (P_2)_A = (P_3)_A.$$

The derivation of a formula for P_A for P a triangle centre sets out from the trilinear coordinates (TLC) of P. An extensive listing of TLC is available in [6]. Background information on TLC, as well as good discussion of them for triangle centres (2) is found in [3–6, 13, 15]. While a great deal of detailed information about TLC is contained in the old book [11], it may be found difficult to locate a hard copy, and too long to download from the web. If P has TLC $x_P : y_P : z_P$, $x_P = f_P(a,b,c)$ defines the triangle centre function, TCF, of P, to within a scale factor which depends on a, b, c only through the invariants s, q, t. Then [15], the result

$$\mathbf{OP} = \frac{ax_P\mathbf{OA} + by_P\mathbf{OB} + cz_P\mathbf{OC}}{ax_P + by_P + cz_P}, \tag{29}$$

relates the TLC of P to the coefficient P_A of $\mathbf{OA}$ in (29). By chosing the scale factor in question suitably, results for the TLC of P lead easily to expressions for P_A. For example, when P is the circumcentre C', $x_{C'} = \gamma = \cos A$, yields $C'_A = \frac{1}{2}\Gamma_A$, where Γ_A is defined by (14), via the identity

$$a\cos A + b\cos B + c\cos C = \frac{2\Delta}{R}. \tag{30}$$

The results for the points (3) follow directly from their definitions given in Section 4.1 below, and data for the points (2) from Table 1. Formulas for the P_A etc. for the centres of Table 2 are derived in Sections 3.2 to 3.8.

In Tables 1 and 2, the column headed TCF gives the first of the TLC of P, that is, x_P. The scale factor $w(P)$ in the sixth column converts general TLC $(x_P : y_P : z_P)$ to exact TLC $(X_P : Y_P : Z_P) = w(P)(x_P : y_P : z_P)$. Explicitly

$$w(P)(ax_P + by_P + cz_P) = 2\Delta = (aX_P + bY_P + cZ_P), \tag{31}$$

and

$$w(P)x_P = X_P = \frac{2\Delta}{a}P_A. \tag{32}$$

3.1 A basis for the quantities P_A for triangle centres

Motivated by (22) for Λ_A and (23) for Γ_A, it is shown here that, and how, all the P_A of Tables 1 and 2 can be expanded in terms of the basis $\mathcal{B}_A = \{1, a, a^2\}$, *i.e.* P_A can be expressed in the form

$$P_A = c_1 + c_2 a + c_3 a^2 \tag{33}$$

where the coefficients are invariant under permutations of a, b, c. This is important since it offers a systematic method of proof of any linear relationship amongst the P_A, and hence amongst the position vectors **OP** of triangle centres. For the triangle centres of Table 1, since the claim is true for Γ_A and for Λ_A, it is evidently true for all the rest, and for K, M, Σ, Σ' as well.

For the triangle centres B', A', Q, D, E, more work is needed. Noting (26) and (27) for P_A and their obvious analogues for the TLC of P, it follows that b and c must enter x_A and P_A in forms such as

$$\begin{aligned} b + c &= 2s - a \\ bc = \frac{t}{a} &= a^2 - 2as + q \qquad\qquad (34)\\ b^2 + c^2 &= (a^2 + b^2 + c^2) - a^2 = 2(2s^2 - q) - a^2 \\ \cos A &= \frac{a}{t}(2s^2 - q - a^2) \\ \cos B \cos C &= -\cos A + \frac{t}{4aR^2} \\ (b - c)^2 &= -3a^2 + 4sa + 4(s^2 - q). \qquad\qquad (35) \end{aligned}$$

Upon using results such as those just quoted on the raw input from [6] for the x_A for B', A', Q, it appears at first glance that one can conclude only that the corresponding P_A are polynomials of degrees $4, 5, 7$ in a with invariant coefficients.

However any polynomial in a can be reduced to one of degree at most two. For this purpose, note that a, b, c are the roots of the cubic equation

$$\begin{aligned} 0 &= (\lambda - a)(\lambda - b)(\lambda - c) \\ &= \lambda^3 - 2s\lambda^2 + qa - t, \qquad\qquad (36) \end{aligned}$$

so that

$$a^3 = 2sa^2 - qa + t, \tag{37}$$

which allows a^4 etc., also to be expanded as (33) calls for. Eq. (37) has already been used in (34).

Once ax_A has been obtained explicitly in the form (33), evaluate the denominator $ax_A + by_A + cz_A$ of P_A, using (5) to treat the a^2 type terms, and (8) for the a type terms.

3.2 The Brocard midpoint $B' = X(39)$

This is the midpoint of the join of the first and second Brocard points of ΔABC, [1], page 263.

The entry $x_{B'} = a(b^2 + c^2)$ in [6] leads to two alternative expressions for $ax_{B'}$

$$\begin{aligned} ax_{B'} &= a^2(a^2 + b^2 + c^2) - a^4 = a^2(4s^2 - 2q) - a^4 \qquad\qquad (38)\\ &= (b^2c^2 + c^2a^2 + a^2b^2) - b^2c^2 = (q^2 - 4st) - b^2c^2, \qquad\qquad (39) \end{aligned}$$

where (8) and (9) have been used. The right side of (38) may be reduced to a quadratic in a, using (37) twice. Then evaluation of the denominator in B'_A leads to

$$B'_A = \frac{-qa^2 + 2s(ua - t)}{2(q^2 - 4st)} = \frac{(2s^2 - q)a^2 + 4r^2s^2\Gamma_A}{2(q^2 - 4st)}, \tag{40}$$

where (23) has been used.

It is convenient for Section 5.4 to pass similarly from (39) to

$$B'_A = \frac{1}{2} - \frac{b^2c^2}{2(q^2 - 4st)}. \tag{41}$$

3.3 The centre $Q = X(970)$ of the Apollonius circle

The Apollonius circle, in the specific sense used in the present context [5–8], is the circle which lies outside all three escribed circles of ΔABC and touches each one of them. Its centre is the Apollonius point Q.

For Q, the long expression quoted in [6] can be reduced to the degree six polynomial

$$\begin{aligned} x_Q &= a^4(2s-a)^2 + (a^2q - 3at)(4s^2 - 2q - a^2) \\ &\quad +2sa^5 - 2sa(a^4 + b^4 + c^4), \end{aligned} \tag{42}$$

and (10) is available. Using computer algebra to grind ax_Q to a quadratic in a leads to

$$ax_Q = 8s^2\left[s(s^2+r^2)a^2 + t(s^2-r^2) - ua(s^2-r^2)\right]. \tag{43}$$

Note the occurrence of quantity $(ua-t)$ in (40), (43) and in the expression (23) for Γ_A. The latter observation leads directly to

$$ax_Q = (16r^2s^3)\left[a^2 - (s^2-r^2)\Gamma_A\right]. \tag{44}$$

Now the identity (30) expedites calculation of the denominator of Q_A, so that (44) gives rise to

$$Q_A = \frac{(s^2-r^2)\Gamma_A - a^2}{8Rr}. \tag{45}$$

3.4 The Apollonius point $A' = X(181)$

Let $\hat{A}, \hat{B}, \hat{C}$ be the points where the Apollonius circle, see Section 3.3, touches the three escribed circles of ΔABC. Then $\hat{A}A, \hat{B}B, \hat{C}C$ are concurrent at the Apollonius point A'.

The expression for $x_{A'}$ in Table 2 is obtained from the entry in [6] for $A' = X(181)$. Upon use of (34) it gives rise to

$$ax_{A'} = a(2s-a)^2\left[as(a-s)+t\right]. \tag{46}$$

Continuing as in Sections 3.2 and 3.3 leads to the result

$$A'_A = \frac{s(s^2+r^2)a^2 + t(s^2-r^2) + a(st - qs^2 + qr^2)}{t(s^2-3r^2)}. \tag{47}$$

Note that the first two terms inside the square brackets in (43) coincide with the first two terms of the numerator in (47).

3.5 The Feuerbach point $F = X(11)$

Results quoted in [6] provide two expressions for ax_F namely

$$a\left[1-\cos(B-C)\right], \quad \text{and} \quad (s-a)(b-c)^2. \tag{48}$$

Each of these can be shown to be compatible with the use of $ax_F = (s-a)a^2 - 4r^2s$. The last two entries for F in Table 2 follow easily from the first expression given for ax_F.

3.6 The centres on the line $K(98)$ through G and B'

The points $D = X(76)$ and $E = X(194)$ are known [6] to lie on the central line $K(98)$. They are of interest here because Section 5.3 shows that the set $EB'GD$ is Eulerian.

It follows from [6] that the TLC of D may be chosen so that $x_D = b^3c^3$, and hence, using (9), that

$$D_A = \frac{b^2c^2}{q^2-4st}. \tag{49}$$

Similarly

$$E_A = 1 - \frac{2b^2c^2}{q^2 - 4st}. \tag{50}$$

The point $D = B_3$ is often called the third Brocard point, although it has not been shown to have any geometrical relationship to the first and second Brocard points B_1 and B_2. The points B_1 and B_2 have TLC which may be chosen in the forms

$$\frac{1}{ab^2} : \frac{1}{bc^2} : \frac{1}{ca^2} \quad \text{and} \quad \frac{1}{ac^2} : \frac{1}{ba^2} : \frac{1}{cb^2}, \tag{51}$$

which reveal the fact that they are not triangle centres, since they do not satisfy (27). However these results have suggested that a point with TLC $\frac{1}{a^3} : \frac{1}{b^3} : \frac{1}{c^3}$ makes up a set of three points to be seen in natural association with B_1 and B_2. Such a point is the point D, which is of course a triangle centre, and has been found here, see section 5.4, to belong to the Eulerian set which contains also the Brocard midpoint B'.

3.7 The isogonal conjugate $M' = X(57)$ of the mittenpunkt M

Replacing the expression in [6] for $x_{M'}$ by

$$x_{M'} = b'c', \tag{52}$$

leads to

$$ax_{M'} = sa^2 - as^2 + t = s(b'c' - r^2), \tag{53}$$

using (15). The entry for M'_A in Table 2 now follows with the aid of

$$b'c' + c'a' + a'b' = r(r + 4R), \tag{54}$$

to evaluate the denominator.

The idea that one triangle centre is the isogonal conjugate of another is discussed in a general context in [1], [16]. If P_1 and P_2 are isogonal conjugates, their TLC can be chosen so that $x_{P_1}x_{P_2} = 1$. For example, $x_M = a'$, and $x_{M'} = \frac{1}{a'}$ is equivalent to the entry for M' in Table 2. Also K and G are isogonal conjugates.

An instructive geometric discussion of the latter fact is implicit in Exercises 1, 2 3, 6 on page 201 of [17]. This book is furthermore a valuable source of trigonometric identities, especially those needed in the passage from ax_P to P_A for a triangle centre P.

3.8 The Soddy centres $\Sigma = X(176)$ and $\Sigma' = X(175)$

These are the centres Σ, Σ' of the inner and outer Soddy circles [13], which are the two solutions of the four coin problem [13, 18, 19]. In [6, 13] the results

$$x_\Sigma = 1 + \frac{\Delta}{aa'}, \quad x_{\Sigma'} = 1 - \frac{\Delta}{aa'} \tag{55}$$

are given. To simplify ax_Σ use

$$\frac{\Delta}{a'} = \frac{b'c'}{r}. \tag{56}$$

Again with the aid of (54) to evaluate the denominators, the entries in Table 2 for Σ_A and Σ'_A follow.

4 Geometrical considerations

4.1 About the points J, T, W, U, V of Table 1

The section begins with definitions, and follows with theorems, which explain in part the motivation for including the points (3) in the present work.

Definition. J is the midpoint of MG'.

Definition. T is defined by the requirement that C' is the midpoint of MT.

Definition. W is the midpoint of HI.

Definition. U is the midpoint of $N'L$.

Definition. V is defined by the requirement that H is the midpoint of $N'V$.

Theorem 1. The lines MC', HJ, LG' are concurrent at the point T, and J, C', T are the midpoints of HT, MT, LG'.

Theorem 2. The lines $HS, IC', N'L$ are concurrent at the point U, and S, C', U are the midpoints of $UH, IU, N'L$.

Theorem 3. The lines $UW, N'H, LI$ are concurrent at the point V, and W, H, I are the midpoints of $UV, N'V, LV$.

Theorem 4. The lines $HC', TG', N'U$ are concurrent at the point L, and C', T, U are the midpoints of $HL, G'L, N'L$.

All of these theorems admit easy proofs using data from Table 1. For example, for the first one, the vector identities

$$\mathbf{OT} = 2\mathbf{OC}' - \mathbf{OM} = 2\mathbf{OJ} - \mathbf{OH} = (\mathbf{OL} + \mathbf{OG}')/2, \tag{57}$$

contain the entire content of Theorem 1.

Sections 5.1 to 5.3 contains a further concurrency result, involving the lines NS, IA', and KB'. A concurrency statement involving the point $X(517)$ at infinity, shared by the three parallel lines HN', IC', and NS is discussed in Section 4.3, together with selected similar results.

4.2 Eulerian lines

Not only are Eulerian lines of interest in their own right, but they also give rise to a good body of geometrical relationships.

It is well-known that $HNGC', LC'GH$ and $N'SGI$ are Eulerian. The ratio $G'G : GM$ too is well-known [6], so that the obvious introduction of J, the midpoint of MG', makes $G'JGM$ Eulerian. Further from the definitions of U, V, W, it follows also that $VWGU$ is Eulerian. Section 5.4 identifies yet another one.

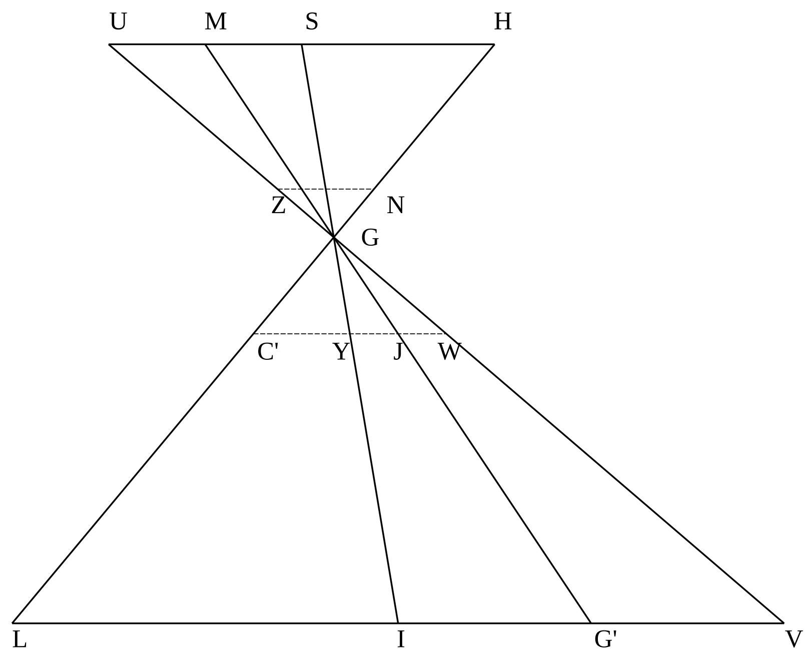

Fig. 3: Eulerian lines, and parallel transversals.

Much more can be said with the aid of Fig. 3 and Fig. 4. Each one of these is a *schematic* diagram, not pretending to be accurate in the context of Figs. 1 and 2. Each one, however, in its essential features, can be seen to be compatible with Figs. 1 and 2, and designed solely to make clear the geometry of its own context, and no more.

The lines UV, MG', SI, HL in Fig. 3 all pass throught the centroid G, and the Eulerian nature of MGG', SGI, HGL means that the lines $UMSH, LIG'V$ are parallel. The points C', N on $LC'GNH$ are marked in Fig. 3. The line $C'YJW$ has been drawn through C' parallel to UH and LV to meet SI, MG', UV at Y, J, W. This is compatible with the facts that J by definition is the mid-point of MG', and that W is the midpoint of UV by Theorem 4. The line NZ through N parallel to $UH, C'W, LV$ meets UV at Z.

Since $K(2) = LC'GNH$ is doubly Eulerian, with ratios $HN : NG : GC : CL = 3 : 1 : 2 : 6$, it is clear the same ratios apply along the line $K(103) = UZGWV$. Also, while Fig. 3 indicates that $SGYI$ is Eulerian with $SG : GY : YI = 2 : 1 : 3$, the point N' could not conveniently be included in Fig. 3. However, it is well-known that [6] $N'SGI$ is Eulerian, so that $K(10) = N'SGYI$ is doubly Eulerian.

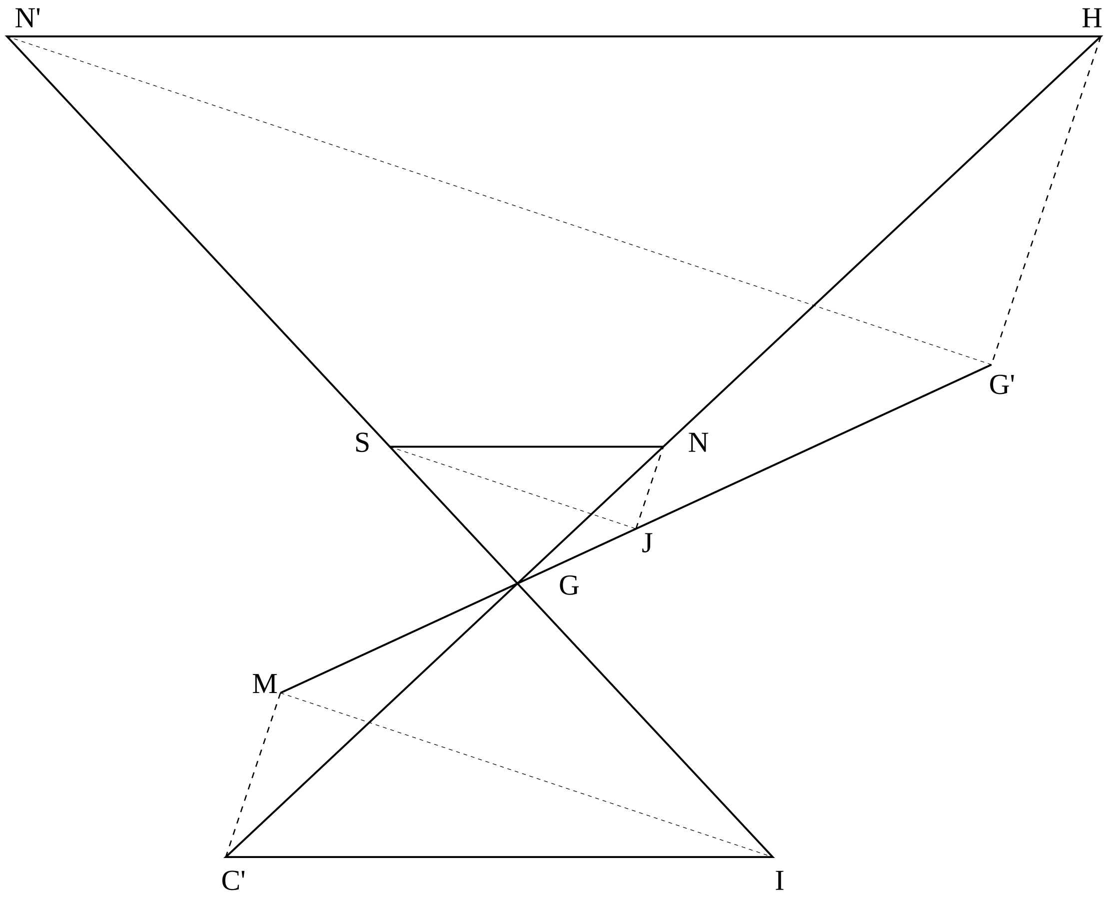

Fig. 4: Eulerian lines, and similar triangles.

The doubly Eulerian nature of the lines $LC'GNH$ and $VWGZU$ implies not only that $LV, C'W, ZN, UH$ are parallel but also that

$$\mathbf{LV} = 2\,\mathbf{UH} = 4\,\mathbf{C'W} = 8\,\mathbf{ZN}. \tag{58}$$

Similarly from $G'JGM$ and $IYGS$, it follows that

$$\begin{aligned} \mathbf{G'V} &= 2\,\mathbf{UM} = 4\,\mathbf{JW} && (59)\\ \mathbf{IG'} &= 2\,\mathbf{MS} = 4\,\mathbf{YJ} && (60)\\ \mathbf{LI} &= 2\,\mathbf{SH} = 4\,\mathbf{C'Y}, && (61) \end{aligned}$$

consistently with (58).

Also, using results from Section 5.7 in the last line,

$$\begin{aligned} UM : MS : SH &= WJ : JY : YC' \\ &= VG' : G'I : IL \\ &= 4R : r : r + 4R. \end{aligned} \tag{62}$$

Further when N' is included, the doubly Eulerian nature of $N'SGYI$, it follows that

$$\mathbf{N'L} = 2\,\mathbf{HI} = 4\,\mathbf{SC'} = 8\,\mathbf{NY}. \tag{63}$$

While the geometrical definition of Y, Z, like that of J, is clear, as is the motivation for considering them here, no one of them has been identified with any triangle centre listed in [5].

Turn next to Fig. 4. This highlights the Eulerian lines $HNGC'$, $N'SGI$, $G'JGM$, so that HN', NS, IC' are parallel, as are $N'G', SJ, MI$, and also $C'M, JN, G'H$. Hence

$$\begin{aligned} \mathbf{N'H} &= 2\,\mathbf{IC'} = 4\,\mathbf{SN} \qquad (64)\\ \mathbf{G'H} &= 2\,\mathbf{C'M} = 4\,\mathbf{JN} \qquad (65)\\ \mathbf{G'N'} &= 2\,\mathbf{IM} = 4\,\mathbf{JS}. \qquad (66)\end{aligned}$$

From the pair $\Delta C'GN'$ and ΔNGI, and the pair $\Delta C'GG'$ and ΔNGM of similar triangles, there follows

$$\begin{aligned} \mathbf{C'N'} &= 2\,\mathbf{IN} \qquad (67)\\ \mathbf{C'G'} &= 2\,\mathbf{MN}. \qquad (68)\end{aligned}$$

All the results obtained above may also be shown to be true using information contained in Table 1.

Just one more geometrical item ... All of

$$HULI,\ HMLT,\ N'HIU, MHG'T, ... \qquad (69)$$

are parallelograms. The diagonals of the last named bisect each other at the point J.

4.3 Parallel lines and points at infinity

In view of the role of central lines which are parallel in Section 4.2, it seems to be natural to give attention to the associated points at infinity. Two, or more, central lines are parallel, iff they have a common point at infinity, itself a triangle centre. Points at infinity are listed under the line $K(1)$ at infinity in [6].

Here are some examples, showing which are the points at infinity lying on certain of the central lines studied here. The easiest case to treat shows that $X(517)$ is the point at infinity [6] on the line $K(11)$ through I and C'. To prove this, write the TCF of $X(517)$ taken from [6] in the form

$$x_{517} = -1 + \beta + \gamma = -1 - \alpha + (1 + \frac{r}{R}) = \frac{r}{R} - \cos A, \qquad (70)$$

so that

$$x_{517} = \frac{r}{R} x_I - x_{C'}. \qquad (71)$$

That $X(517)$ is indeed a point at infinity is reflected by the result

$$a x_{517} + b y_{517} + c z_{517} = 0. \qquad (72)$$

Similarly taking

$$x_{N'} = \sin B \sin C - \frac{r}{R}, \quad x_S = \sin B \sin C - \frac{r}{2R}, \qquad (73)$$

it follows that

$$x_{517} = x_H - x_{N'} = x_N - 2x_S, \qquad (74)$$

so that $X(517)$ lies on the lines $K(23) = HN'$and on $K(35) = NS$. The fact that IC', NS, and HN' are parallel transversals of the Eulerian lines $HNGC'$ and $N'SGI$ has already been noted in Section 4.2.

Similarly $X(515)$ is the point of infinity on the lines $K(12) = IH$, $K(25) = C'S$, and $K(67) = N'L$, which are parallel transversals of the lines $LC'GH$, and $N'SGI$. Converting the result in [6] into the form

$$x_{515} = \cos B \cos C - \frac{r}{2R}, \qquad (75)$$

and using (73) leads to the results

$$\begin{aligned} x_{515} &= x_H - \frac{r}{2R}x_I \\ &= x_S - x_{C'} \\ &= (x_{N'} - x_L)/2, \end{aligned} \tag{76}$$

which confirm that $X(515)$ lies on the three lines indicated. Also the analogue of (72) for $X(515)$ is easily seen to hold.

A third example involves the parallel transversals of $IGSN'$ and $EB'GD$. In this case $X(730)$ is the point at infinity on $K(225) = ID$, $K(125) = SB'$, and the line $N'E$. Proof uses the input

$$\begin{gathered} ax_I = a, \qquad ax_S = 2s - a, \qquad ax_{N'} = s - a \\ ax_D = b^2c^2, \qquad ax_E = -2b^2c^2 + k, \qquad ax_{B'} = (-b^2c^2 + k)/2 \\ ax_{730} = 2sb^2c^2 - ka, \end{gathered} \tag{77}$$

where $k = q^2 - 4st$, to justify writing

$$\begin{aligned} x_{730} &= 2sx_D - kx_I \\ &= -sx_E + kx_{N'} \\ &= -2sx_{B'} + kx_S. \end{aligned} \tag{78}$$

4.4 More about the points (3)

The content of Section 4.1 indicates that U, V, W are the intersections of significant central lines, as can be seen also in Fig. 1. Hence, reference to information in the Table of Central Lines [6] allows the identifications

$$U = X(40), \quad W = X(946), \quad V = X(962) \tag{79}$$

to be made.

The points J and T do not appear to coincide with any listed triangle centre.

Since it is far from easy to relate the entries in [6] for the TLC of these centres to the simple manageable entries for U, W, V in Table 1, some detail is provided.

From [6] take the entry x_U for the triangle centre function for $U = X(40)$ in the form

$$-ab'c' + bc'a' + ca'b' = 2s\left[aa' - r^2 - 2rR\right]. \tag{80}$$

Use (37) to write ax_U in the form called for by (33). Continuing as in Section 3.2 for the Brocard point, find first that

$$2r^2sU_A = -sa^2 + a(u - r^2) - t, \tag{81}$$

and then, using (23) to eliminate the a^2 term here, confirm that the Table 1 entry for U_A is correct.

From [6], use the entry for $W = X(946)$ there to obtain

$$\begin{aligned} ax_W &= a^3(b + c) + (b - c)^2(a^2 - ab - ac - 2bc - b^2 - c^2) \\ &= a^3(2s - a) + (4s^2 - 4q + 4sa - 3a^2)(a^2 + 2sa - 4s^2), \end{aligned} \tag{82}$$

reducible on the basis of (37), to

$$ax_W = 8s\left[sa^2 - a(u - r^2) + 2sr(r + 2R)\right]. \tag{83}$$

This turns eventually but routinely into a result which agrees with the Table 1 result for W_A.

From [6], use the entry for $V = X(962)$ to obtain

$$\begin{aligned} ax_V &= a^4 + 2a^3(b + c) - 4a^2bc - (b - c)^2(b + c)(2a + b + c) \\ &= a^4 + 2a^3(2s - a) - 4at - (4s^2 - a^2)(4s^2 - 4q + 4sa - 3a^2), \end{aligned} \tag{84}$$

which reduces to

$$ax_V = 16s\left[s(q-s^2)+sa^2-a(u-r^2)\right], \tag{85}$$

and leads eventually to agreement with the Table 1 entry for V_A.

Given the identifications just made, it can be seen that $VWGU$ is $K(103)$ in [6]. These are the only four listed points on it. There is of course also a point at infinity on $K(103)$. The expression $x_U - 2sr^2$ for its TCF can be shown to be correct. Use of the entries in Table 1 for V, W, G, U confirms the fact deduced in Section 4.2 by geometrical means that $VWGU$ is Eulerian.

5 Ratios of distances between triangle centres on central lines

5.1 The Brocard axis $K(3) = KB'C'Q$

Using data from the Tables for K and C' in (40) gives rise to the result

$$\begin{aligned} 2(q^2-4st)B'_A &= (2s^2-q)a^2+4r^2s^2\Gamma_A \\ &= 2(2s^2-q)^2K_A+8r^2s^2C'_A, \end{aligned} \tag{86}$$

thence to

$$\begin{aligned} (q^2-4st)\,(B'_A-K_A) &= 4r^2s^2\,(C'_A-K_A) \quad \text{and} \\ (q^2-4st)\,\mathbf{KB'} &= 4r^2s^2\,\mathbf{KC'}, \end{aligned} \tag{87}$$

and hence to

$$(2s^2-q)^2\,\mathbf{KB'} = 4r^2s^2\,\mathbf{B'C'}. \tag{88}$$

Eq. (88) confirms the collinearity of $KB'C'$, indicates that this is their order along $K(3)$ and provides the formula for the ratio

$$\frac{KB'}{B'C'} = \left[\frac{2rs}{2s^2-q}\right]^2. \tag{89}$$

To see that the Apollonius centre also lie on $K(3)$, refer to (45). This immediately yields

$$8Rr\,Q_A = 2(s^2-r^2)C'_A - 2(2s^2-q)K_A, \tag{90}$$

from which there follows

$$4Rr\,\mathbf{KQ} = (s^2-r^2)\,\mathbf{KC'} \tag{91}$$

This confirms collinearity, gives the order $KC'Q$ along $K(3)$, and yields the formula

$$\frac{KC'}{C'Q} = \frac{4Rr}{2s^2-q}. \tag{92}$$

Results equivalent to (91) are found as Proposition 5 of [8], and Proposition 3 of [7].

The results (89) and (92) may be combined to give

$$KB' : B'C' : C'Q = \frac{4r^2s^2}{2s^2-q} : 2s^2-q : \frac{q^2-4st}{4Rr}. \tag{93}$$

5.2 The line $K(35) = NSQ$

The line through the centre N of the nine-point circle and the Spieker centre S passes through Q. This known fact [6] can be seen in Fig.2. Use of (23) to eliminate the a^2 term of (45) for Q_A yields the result

$$8Rrs\,Q_A = s(s^2+r^2)\Gamma_A - (ua-t). \tag{94}$$

Now use of $\Gamma_A = 2 - 4N_A$ and $a = 2s - 4s\,S_A$ leads, after use of (7) to make some cancellations, to

$$2Rr\,Q_A = u\,S_A - (s^2 + r^2)N_A. \tag{95}$$

Hence, using (7) again,

$$2Rr\,\mathbf{SQ} = (r^2 + s^2)\,\mathbf{NS}, \tag{96}$$

and

$$\frac{NS}{SQ} = \frac{2rs}{r^2 + s^2}. \tag{97}$$

The point W also lies on $K(35)$. Since N is defined to be the midpoint of SW, $\mathbf{NS} = \mathbf{WN}$ extends (96).

In view of the role of the Spieker centre S in this, the next subsection and elsewhere, an alternative view of it to the well-known geometrical one is mentioned. This states that S is the centre of mass of a wire of uniform density in the shape and position of the triangle ΔABC. See [20], exercise 5c.8, or [1]. page 249.

5.3 The line $K(600) = A'IQ$

The relationship of the numerators of (47) for A'_A and (43) for Q_A has already been emphasised. As a consequence of it, it follows that

$$t(s^2 - 3r^2)A'_A + 4r^2 tQ_A = 2Rr(s^2 + r^2)a = t(s^2 + r^2)I_A, \tag{98}$$

whence

$$(s^2 - 3r^2)A'_A + 4r^2\,Q_A = (s^2 + r^2)I_A, \tag{99}$$

and

$$(s^2 + r^2)\,\mathbf{IA'} = 4r^2\,\mathbf{QA'}. \tag{100}$$

However, to proceed, it must be shown that $s^2 - 3r^2$ is always positive. First note

$$0 < (a^2 + b^2 + c^2)/2 = s^2 - r(r + 4R).$$

Using also (117) from Section 5.6, it follows that

$$s^2 > r(r + 4R) > \sqrt{3}rs, \tag{101}$$

so that $s > \sqrt{3}r$ and $s^2 > 3r^2$. Return then to (100), which indicates that Q and I lie on the same side of A' on $K(600)$, since both the coefficients there are positive. But $(s^2 + r^2) > 4r^2$, so that the order of the points along $K(600)$ is QIA'. Hence (100) yields the ratio

$$\frac{QI}{IA'} = \frac{s^2 - 3r^2}{4r^2}. \tag{102}$$

5.4 The line $K(98) = EB'GD$

Using (39) for B' in terms of b^2c^2, (49) for D_A, and (50) for E_A, it is very easy to find

$$3G_A = 2B'_A + D_A, \tag{103}$$

and

$$3G_A = 2D_A + E_A, \tag{104}$$

and to prove that $EB'GD$ is Eulerian, *i.e.*

$$EB' : B'G : GD = 3 : 1 : 2. \tag{105}$$

Now treat the Eulerian lines $K(98) = EB'GD$ and $N'SGI = K(10)$, in the same way as Section 4.2 treats the lines $HNGC'$ and $N'SGI$, using Fig. 4. It follows that

$$\mathbf{EN'} = 2\,\mathbf{ID} = 4\,\mathbf{B'S}. \tag{106}$$

That the lines involve here are parallel has been noted in Section 4.3 as a consequence of the fact that they share the point $X(730)$ at infinity.

5.5 The line $K(16) = MGJG'M'$

For the isogonal conjugate M' of the mittenpunkt M, evaluation of M'_A, using (16) for $b'c'$, and then the Table 1 entries for G and M, leads directly to

$$(2R - r)M'_A = 6R\, G_A - (r + 4R)M_A, \tag{107}$$

so that

$$\frac{MG}{GM'} = \frac{2R - r}{r + 4R}. \tag{108}$$

From the result $C'I^2 = R^2 - 2rR$ given in [1], page 186, and in [2], page 35, it follows that $R > 2r$, so that $2R > r$. It has already been noted, in Section. 4.2, that $G'JGM$ is Eulerian. Hence

$$MG : GJ : JG' : G'M' = 2 : 1 : 3 : \frac{6r}{2R - r}. \tag{109}$$

The line $MGJG'M'$ is thus not doubly Eulerian.

5.6 The line $K(11) = UC'IM'$

Again, the use of (16) for $b'c'$ to evaluate M'_A, leads directly to

$$(2R - r)M'_A = (r + 2R)\, I_A - 2r\, C'_A, \tag{110}$$

and

$$(2R + r)\, \mathbf{IM'} = 2r\, \mathbf{C'M'}. \tag{111}$$

This implies the ordering $C'IM'$ on $K(11)$, and hence

$$\frac{C'I}{IM'} = \frac{2R - r}{2r}. \tag{112}$$

The fact that $U = X(40)$ lies on $K(11)$ follows from the statement in Theorem 4 that C' is the midpoint of IU.

5.7 The line $K(9) = G'IL$

This is called the Soddy line in [6] because the centres $\Sigma = X(176)$ and $\Sigma' = X(175)$ of the inner and outer Soddy circles lie on it, which is why they are included in Table 2. The points T and V of Table 1 also lie on the line.

It is known from theorems 1 and 3 that $LT = TG'$ and $LI = IV$, so that also $G'V = 2\, TI$. To fix the ratios of the lengths of all the intervals between points along $LTIG'V$ one more result is needed. For this purpose, there is the identity

$$(r + 4R)G'_A = 2(r + 2R)I_A - r\, L_A, \tag{113}$$

which can be checked using data from Table 1. Eq. (113) implies

$$(r + 4R)\, \mathbf{IG'} = r\, \mathbf{LI}. \tag{114}$$

A result related to (114) can be found in [13] on page 328. Hence

$$LT : TI : IG' : G'V = (r + 2R) : 2R : r : 4R. \tag{115}$$

There is no choice of any subset of four points along $LTIG'V$ which is Eulerian for any possible values of r and R. To see this recall the fact, noted in Section 5.4, that $R > 2r$. Actual lengths follow from (115), since it is known [12] that

$$LI^2 = (r + 4R)^2 - 3s^2, \tag{116}$$

so that

$$(r + 4R) > \sqrt{3}s. \tag{117}$$

Turning next to the inner Soddy centre $\Sigma = X(176)$, the result [14]

$$(r + 4R)\ \mathbf{\Sigma G'} = 2s\ \mathbf{I\Sigma} \tag{118}$$

shows that Σ lies on $K(9)$ between I and G', and gives the ratio $I\Sigma : \Sigma G'$.

When the outer Soddy centre $\Sigma' = X(175)$ is considered, inspection of Σ'_A shows that its denominator can take on either sign. Although

$$(r + 4R)\ \mathbf{G'\Sigma'} = 2s\ \mathbf{I\Sigma'}, \tag{119}$$

is true for either sign, it needs to be interpreted in subtly different ways in the two cases.
Case (i): $2s > (r + 4R) > \sqrt{3}s$. Eq. (119) implies that I and G' are on the same side of Σ', with I closer to it than G'. Thus the order of points on $K(9)$ is $G'I\Sigma'$, and hence $G'\Sigma I\Sigma'$. Also

$$(r + 4R)\ \mathbf{G'I} = (2s - r - 4R)\ \mathbf{I\Sigma'}, \tag{120}$$

consistently. From use of (118) and (119), it follows that

$$\frac{G'\Sigma}{\Sigma I} = \frac{G'\Sigma'}{I\Sigma'} = \frac{2s}{r + 4R}. \tag{121}$$

Thus [13], page 327, $G'\Sigma I\Sigma'$ is a harmonic range, a concept discussed, for example, on [2],page 65.
Case (ii): $2s < (r + 4R)$. This time (119) gives the order $I\Sigma G'\Sigma'$ on $K(9)$, implies

$$2s\ \mathbf{IG'} = (r + 4R - 2s)\ \mathbf{G'\Sigma'}, \tag{122}$$

again consistently, and provides proof that $I\Sigma G'\Sigma'$ is a harmonic range.

When $2s = (r + 4R)$, a formula for the radius σ' of the external (outer) Soddy circle, known to Descartes [18], and found in the form

$$\frac{1}{\sigma'} = \frac{2}{r} - \left(\frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'}\right), \tag{123}$$

in [13], has right side equal to zero, so that σ' is infinite [14]. This indicates that Σ' is the point at infinity on $K(9)$, and the analogue of (72) confirms this. Thus, when $2s = (r + 4R)$, the right sides of (120) and (122) are undefined.

5.8 The line $K(26) = UMSH$

In [21], it was proved using REDUCE that M, S, H are collinear. To prove even this otherwise involves a little non-trivial trigonometry. But using the methods employed throughout this paper, it is a routine matter to prove that

$$\frac{HS}{SM} = \frac{r + 4R}{r} \tag{124}$$

Considerable simplification of eqs. (5) and (8) of [22] provides another derivation of (124), while a nice geometrical proof, and more, is to be found in Section 4.2.

5.9 The line $K(14) = MIK$

It is straightforward to show that

$$(q - s^2)\ \mathbf{KM} = s^2\ \mathbf{KI}, \tag{125}$$

and hence, use data from Tables 1 and 2, to confirm the result [22]

$$\frac{KI}{IM} = \frac{q - s^2}{2s^2 - q}. \tag{126}$$

5.10 The line $K(13) = NIF$

The collinearity of the three points named follows directly from their TLC. The expressions quoted for N_A, I_A, F_A then yield

$$(R - 2r)\mathbf{OF} = R\,\mathbf{OI} - 2r\,\mathbf{ON}, \tag{127}$$

so that

$$\frac{NF}{IF} = \frac{R}{2r}. \tag{128}$$

Since the nine-point and inscribed circles have centres N and I, and radii $\frac{R}{2}$ and r, a well-known property of the Feuerbach point, that it, is the point where the two circles touch, emerges [1], page 200.

The Fuhrmann centre $F' = X(355)$ also lies [6] on the line $K(13)$. By writing the result from [6] for its TLC in the form $x_{F'} = \frac{r}{R} - \cos(B - C)$, so that

$$F'_A = \frac{a}{2rs}(R\cos(B - C) - r), \tag{129}$$

it follows that N is the mid-point of IF'. This locates F' precisely on $FINF'$ at a distance $F'N = NI = \frac{1}{2}R - r$ beyond N.

It follows also that F' is the mid-point of $N'H$. The Fuhrmann circle is decribed in [1], page 228, and it is shown there that $N'H$ is a diameter of this circle. Hence, using (64), the radius of the Fuhrmann circle is

$$\frac{1}{2}HN' = IC' = \sqrt{R(R - 2r)} \geq 0. \tag{130}$$

6 About the Figs. 1 and 2

There is evidently an extremely rich structure in the pattern of the triangle centres and central lines of a reference triangle ΔABC. To enable appreciation of this pattern, two accurate diagrams have been prepared. The underlying choice therein of reference triangle is made on the basis of a great deal of experimentation, and in order to avoid such flaws as the near coincidence of important centres. Fig. 1 is designed to display as many features on the scene of the triangle centres of Table 1, as seems reasonable. The bold continuous lines are the Eulerian lines

$$HNGC', \quad N'SGI, \quad G'JGM, \quad (V)WGU. \tag{131}$$

The points K, B' and E are shown simply because they can be without obscuring other features, whereas drawing the line $A'I$ and producing it towards Q is not a good idea since this line makes too small an angle with other displayed lines. Also, the points L, V and the vertices of the reference triangle are not visible in Fig. 1, because they, like Q, lie well outside the bounding box of Fig. 1, as the notations "to L, to V" indicate. To remedy this situation, and also bring A' into the overall picture, Fig. 2 is provided using a smaller scale, with enough detail to make clear the relationships between the two displays.

The facts that
(i) $HNGC'$ and $G'I$ meet in L
(ii) NS, IA', KC' are concurrent at Q
are seen in Fig. 2.

The line $N'SGI$ is included to indicate the locations of S and G in Fig. 2. Also the Brocard midpoint B' lies at the point where the EG portion of the Euler line $EB'GD$ intersects the line $K(B')C'Q$. Further the lines $HNGC'L$ and $K(B')C'Q$ are seen to intersect at C'. Inspection of Fig. 1 suggests that $IN = K(13)$ and $C'N' = K(19)$ are parallel, and (67) confirms this.

7 An afterthought

A project like the one described in this paper never reaches a natural end. Indeed, looking once more at Fig. 2, the question arises: what can be said about the point P' of intersection of $K(3) = KB'C'Q$ and $K(9) = G'IL$? First it is easy to find out [6] that $P' = X(991)$.

The methods followed throughout this paper allow the position vector of P' to be found and identified both as a point on LI and on KC'. Thus

$$qP'_A = uI_A + 2RrL_A = (2s^2 - q)K_A + 2(q - s^2)C'_A, \tag{132}$$

giving the ratios in which P' divides LI and KC'. Checking that this result is compatible with the expression

$$x_{P'} = a\left[a^3(b + c) - a^2(b^2 + c^2 - bc) - a(b + c)(b^2 + c^2) + (b - c)(b^3 - c^3)\right], \tag{133}$$

is not easy. However the factor of (133) in square brackets reduces, with the aid of formulas for invariants given in Sec. 2.1, to $4(q - s^2)a^2$ plus a constant term, invariant under permutations of a, b, c, and explicitly expressible as a function of s, r, R. It follows that the result

$$ax_{P'} = ta^2 - (q - s^2)(ua - t) \tag{134}$$

can be reached with the aid of (37) to treat the a^4 term of $ax_{P'}$. Hence, using

$$(ax_{P'} + by_{P'} + cz_{P'} = -2qsr^2, \tag{135}$$

agreement with (132) is established.

Appendix A: Expressions for distances between triangle centres

A collection of expressions for distances between triangle centres is presented here. To achieve a complete compilation of modest size attention is confined to the centres

$$I, G, C', H, N, G', N', M, S, L \tag{136}$$

of Table I. The first six results

$$C'I^2 = R(R - 2r) \tag{137}$$
$$C'H^2 = 2r^2 + 8Rr + 9R^2 - 2s^2 \tag{138}$$
$$IH^2 = 3r^2 + 4Rr + 4R^2 - s^2 \tag{139}$$
$$9IG^2 = 5r^2 - 16Rr + s^2 \tag{140}$$
$$IN = \frac{R - 2r}{2} \tag{141}$$
$$IL^2 = (r + 4R)^2 - 3s^2 \tag{142}$$

are quoted without proof because they are well-known from texts on trigonometry, such as [17]. They serve as input to the proof of results that follow. Given (138), (140), (142) for $C'H, IG, IL$, no other results for the distances between pairs of points on the lines $C'LGNH, IGSN', LIG'$ need to be displayed. Since $IL = 2HS$, the distance between any two points of MSH can be written down using (73). To ensure that an explicit formula for the distance between any two of the points (136) can be written down on the basis of information readily available in this paper, ten further results are listed.

$$\begin{aligned}
HG'^2 &= -8R(2R-r)\Omega^2 + 4R^2 & (143)\\
N'G'^2 &= 16R(R+r)\Omega^2 - 16Rr & (144)\\
C'G'^2 &= -4r(r+R)\Omega^2 + R^2 & (145)\\
4NG'^2 &= 6r(4R-r)\Omega^2 + R^2 - 8Rr - r^2 & (146)\\
9MG^2 &= (2R-r)(2R+5r)\Omega^2 - r(r+4R) & (147)\\
4LS^2 &= 21r^2 + 64Rr + 64R^2 - 15s^2 & (148)\\
4SG'^2 &= 4(4R+7r)(4R-r)\Omega^2 - r(3r+16R) & (149)\\
MN'^2 &= -(4R^2-r^2)\Omega^2 - r(3r-4R) & (150)\\
4NN'^2 &= 6r^2 - 32Rr + R^2 + 2s^2 & (151)\\
ML^2 &= -(r+6R)(r+10R)\Omega^2 + 3r^3 + 12Rr + 16R^2 & (152)
\end{aligned}$$

In (143) – (152), the abbreviation $\Omega = \dfrac{s}{r+4R}$ has been introduced. Eq. (147) for MG allows distances bewteen points on NGG' to be written down. Beyond this only results quoted in Section 4.2 are needed. A table may be drawn up to verify that all forty-five distances are indeed covered.

Proof of the results (143) – (152) can be given on the basis of the

Theorem : If the point B of the line AC divides it in the ratio $AB : BC = \lambda : \mu$, and X is a point not on the line AC, then

$$(\lambda+\mu)BX^2 + \lambda BC^2 + \mu AB^2 = \lambda CX^2 + \mu AX^2. \tag{153}$$

Proof is a simple exercise in trigonometry that makes two uses of the cosine rule. It is convenient to refer to the result (153) as 'X against ABC'. The results (143) – (146), use the cases H, N', C', N against LIG' of (153). The results (147), (148), use the cases M, L against IGS. Once LS has been treated, S against LIG' yields (149). For MN' and NN', use M against IGN', and N' against LNH. To treat ML via M against $HC'L$, note from (65) that $C'M = HG'/2$.

Result (141) implies that the nine-point circle touches the incircle externally, which is part of Feuerbachs's theorem [1], page 200, as is shown using a different approach in Section 5.10. The results

$$R^2 - C'M^2 = 2R(2R-r)\Omega^2, \quad R^2 - C'G'^2 = 4r(r+R)\Omega^2 \tag{154}$$

give the powers of M, G' with respect to the circumcircle of ΔABC.

The writing of the results in this appendix in terms of s, r, R is of course just one choice. While there are neater ways of presenting a few in terms of say a, b, c or even s, q, t, uniformity is what motivated the choice made.

Various checks on the above results have been performed using different methods.

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960

[2] C. V. Durell, *Modern Geometry*, MacMillan & Co. Ltd., London, 1949.

[3] H. S. M. Coxeter, *Some applications of trilinear coordinates*, Lin. Alg. Appl. **226-228** 375–388 (1995).

[4] M. Longuet-Higgins, *A four-fold point of concurrence lying on the Euler line of a triangle*, Math. Intell. **22** 54–59 (2000).

[5] C. Kimberling, *Central points and central lines in the plane of a triangle*, Math. Mag. **67** 163–187 (1994).

[6] C. Kimberling, Encyclopedia of Triangle Centres,
//http: faculty.evansville.edu/ck8/encyclopedia/ETC.html.

[7] M. R. Stevanovic, *The Apollonius circle and related triangle centres*, Forum Geom. **3** 187–195 (2003).

[8] D. Grinberg and P. Yiu, *The Apollonius circle as a Tucker circle*, Forum Geom. **2** 175–182 (2002).

[9] W. Gallatly, *The modern geometry of the triangle*, second edition, Hodgson, London, 1913.

[10] E. W. Weisstein, Isogonal Mittenpunkt, from MathWorld, A Wolfram Web Resource, http://mathworld.wolfram.com/IsogonalMittenpunkt.html.

[11] W. E. Whitworth, *Trilinear coordinates*, Deighton, Bell, & Co., Cambridge, 1866.

[12] A. Vandeghen, *Soddy Circles and the de Longchamps point of a triangle*, Amer. Math. Monthly **71** 176–179 (1964).

[13] A. Oldknow, *The Euler-Gergonne-Soddy line of a triangle* Amer. Math. Monthly **103** 319–329 (1996).

[14] N. Dergiades, *The Soddy circle*, Forum Geom. **7** 91–97 (2007).

[15] E. W. Weisstein, Trilinear Coordinates, from MathWorld, A Wolfram Web Resource. http://mathworld.wolfram.com/TrilinearCoordinates.html.

[16] E. W. Weisstein, Isogonal Conjugates, from MathWorld, A Wolfram Web Resource, http://mathworld.wolfram.com/ IsogonalConjugate.html

[17] F. G. W. Brown, *Plane trigonometry*, MacMillan & Co. Ltd., London, 1930

[18] René Descartes, *Œuvres*, Publieés par C. Adam and P. Tannery, vol. 4, Paris, 1901.

[19] H. S. M. Coxeter, *The problem of Apollonius*, Amer. Math. Monthly **75** 5–15 (1968).

[20] A. D. Gardiner and C. J. Bradley, *Plane Euclidean geometry*, revised edition, UK Math. Trust, 2012.

[21] R. H. Eddy, *A generalisation of Nagel's Middlespoint*, Elem. Math. **45** 14–18 (1990).

[22] E. W. Weisstein, Mittenpunkt, from MathWorld, A Wolfram Web Resource, http://mathworld.wolfram.com/ Mittenpunkt.html