# Pi Formulas: some smooth stones on the beach of rough numbers


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**Abstract**

This article is about Pi Formulas, infinite series of fractions which sum to simple multiples of Pi. Each such one can be associated with a unique set $S_k$ of $k$-rough numbers, where $k$ is a prime number. Given $S_k$ for any prime $k$, the set $S_{k'}$, where $k'$ is the smallest prime greater than $k$, can be constructed easily. From this it follows that PiFormulas occur in disjoint families. In any family, there is a first member, a series of a least prime number $k_{min}$ , which must be summed from first principles. Then from the series of some $k > k_{min}$ already summed, the series for $k'$ can be summed by a simple algebraic procedure. A good number of Pi Formulas, belonging to a variety of families, and giving results believed to be new, are presented here.



**Keywords**: rough number, infinite series, prime number, modulo multiplication group, totient function, contour integral

**Mathematical Subject Classification**: 11A67, 11B50, 40C10, 40C15


## 1 Introduction

This article presents a number of Pi Formulas believed to be new here, and develops the view that Pi Formulas occur in families defined by association with the family of rough numbers.

Begin by quoting four Pi Formulas

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \int_0^1 \frac{dx}{1+x^2} = \pi/4 \tag{1}$$

$$1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} - \frac{1}{23} + \frac{1}{25} + \cdots = \int_0^1 \frac{1+x^4}{1+x^6} dx = \pi/3 \tag{2}$$

$$\begin{aligned} &(1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} - \frac{1}{23} + \frac{1}{29}) - \\ &(\frac{1}{31} - \frac{1}{37} - \frac{1}{41} + \frac{1}{43} + \frac{1}{47} - \frac{1}{49} - \frac{1}{53} + \frac{1}{59}) + \ldots \\ &= \int_0^1 \frac{(1-x^6)(1-x^{10})(1+x^{12})}{1+x^{30}} dx = \frac{4\pi}{15} \end{aligned} \tag{3}$$

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \frac{1}{23} + \cdots = \int_0^1 \frac{1-x^4}{1-x^6} dx = \frac{\pi\sqrt{3}}{6} \tag{4}$$

Eq. (1) is the famous result of Gregory (1671) and Leibniz (1674). The second is found in [1, eq. (36)]. The third one is proved here in the belief that it is a new result, as are various further results given below, which differ from (3) only in the distribution of signs of the same set of fractions. Eq. (4) is (85) in [2].


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Two good and extensive sources of Pi Formulas are the book of Jolley [2] and [1]. Eq. (2) is not given in [2], although the related result (4), is, and

$$1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \frac{1}{13} - \frac{1}{17} - \frac{1}{19} + \frac{1}{23} + \cdots = \int_0^1 \frac{1-x^4}{1+x^6} dx = \frac{1}{\sqrt{3}} \log(2+\sqrt{3}), \tag{5}$$

not of course a Pi Formula, appears as (84) there. Note also exercise (528) in [3].

The material presented in this article is organised as follows. Since rough numbers and to a lesser extent modulo multiplication groups play an important role in the formalism developd here, Section two provides background information and references for these topics. Section three develops the view that Pi Formulas occur in families in which passage from one member of the family to its successor follows a uniquely defined path. Section four discusses the evaluation of the integrals arising in Pi Formulas such as those quoted above, and also below. Elementary methods, contour integration, and an elegant algebraic procedure, described in this section, are all essential to the systematic accumulation of Pi Formula results. The last of these methods is used to extend the $\Sigma_3$ family of (1) to an $S_{11}$ fourth member, and potentially further. Section five, entitled More Pi Formulas, covers a range of further examples which illustrate themes developed in previous sections, but no systematic survey of the scene is attempted. Section six considers briefly further developments.

# 2 Rough numbers and Modulo Multiplication Groups (MMG)

The $k$-rough numbers [4], defined for all prime numbers $k \geq 2$, are the elements of the set $S_k$, which contains exactly once each positive integer not divisible by any prime less than $k$. Explicitly, showing also reference numbers in [5],

$$\begin{aligned} S_3 =& \{1,3,5,7,9,11,13,15,17\ldots\}, \quad \underline{A005408}, \\ S_5 =& \{1,5,7,11,13,17,19,\ldots\} \quad \underline{A007310}, \\ S_7 =& \{1,7,11,13,17,19,23,\ldots\} \quad \underline{A007775} \\ S_{11} =& \{1,11,13,17,19,23,\ldots\} \quad \underline{A008364}. \end{aligned} \tag{6}$$

The sequences $S_{13}$ and $S_{17}$ have references numbers A008365 and A008366 in [5]. Note that $25, 35, 55, 65, ...$ belong to $S_5$, $49, 77, 119, ...$ to $S_7$, and $121, 143, \ldots$ to $S_{11}$, and so on. Also $S_2 = N$, so that all natural numbers are 2-rough, and $S_1$ is undefined.

There is no entry rough number in the Mathematical Subject Classification, although they feature as noted in [5].

It is indicated in [4], that the $k$-rough numbers were originally introduced in [6], [7, Sec. 2.21]. However it is clear that Euler was well aware of rough number patterns, but, they are dealt briefly with in passing, in the context of studies of the harmonic series, a study in which divergent analogues of $(1-3)$ appear. See [8, p. 58]. It is furthermore implicit there that the $S_k$ are easily constructed in order of increasing $k$ by means of the result

$$S_{k'} = S_k - kS_k, \tag{7}$$

where $k'$ is the smallest prime greater than $k$.

Turning to modulo multiplication groups (MMG), in relation to the $S_k$, define, for each prime $k$, $P_k$ to be the product of all primes less than $k$; in particular

$$P_3 = 2, \quad P_5 = 6, \quad P_7 = 30, \quad P_{11} = 210, \tag{8}$$

and $P_{k'} = kP_k$. Now, if the elements of $S_k$ are reduced modulo $P_k$, the distinct integers obtained provide the elements of a group $G_k = M(P_k)$ of order $g_k = \phi(P_k)$, where $\phi(n)$ is the Euler totient function [9], and the group multiplication law is multiplication modulo $P_k$. The group $M(n)$ is referred to [9, p. 61], [10] as a modulo multiplication group (MMG). For example

$$G_3 = M(2) = \{1\},\ G_5 = M(6) = \{1,5\},\ G_7 = M(30) = \{1,7,11,13,17,19,23,29\} \tag{9}$$

of orders $g_3 = 1,\ g_5 = 2,\ g_7 = 8$. Also $G_{11} = M(210)$ is of order $g_{11} = 48$, and $G_{13} = M(2310)$ is of order $g_{13} = 480$. Note also

$$g_{k'} = \phi(k) g_k = (k-1) g_k, \tag{10}$$

as $k$ is prime.

The first $g_k$ element of each $S_k$ are the elements of $G_k$. The remainder may be written down in successive sets of size $g_k$ obtained by adding $P_k, 2P_k, \ldots$ to elements of the first sets. For example,

$$1, 5; 7, 11; 13, 17; 19, 25; 29, 31; 35, 37; \ldots \tag{11}$$

In all of (1 – 5), in the denominators of the fractions summed over, there is found the complete set of elements of the relevant $S_k$. So (1) is an $S_3$ result. The four that follow in Section one are $S_5, S_7, S_5, S_5$ results.

# 3 Families of $S_k$ results

It is argued here that (1 – 3) provide the $S_k$ results of the $\Sigma_3$ family of such results, for $k = 3, 5, 7$. Results for higher $k$ are obtained further on. Also (4) is the first or $k = 5$ result of a family $\Sigma_5$ of results, independent of the $\Sigma_3$ family. The second member of this $\Sigma_5$ family is found as (26) in Section four.

To justify the view being advanced and account for the statements just made, begin by making some definitions:

i) the functions $f_k(x), \quad k = 3, 5, \ldots$, which are the integrands of the integrals which yield the series of the family $\Sigma_3$, and which are given, for $k = 3, 5, 7$ in (1 – 3)

ii) the functions $g_k(x), \quad k = 3, 5, \ldots$, which play a similar role for the series $\Sigma_5$, with $g_5(k)$ given by (4)

iii) the functions $h_k(x), \quad k = 3, 5, \ldots$, associated with (5). See Sec. 5, last paragraph.

Expansions of the $f_k(x)$ give power series which, upon integration, yield the series on the left side of (1 – 3). As noted above the fractions thereby found contain all the elements of the corresponding $S_k$. When the terms of (1) containing integers divisible by three are excised from it, only the terms of (2) remain, with the signs inherited from (1). Similarly, when the terms of (2) containing integers divisible by five are excised from it, only the terms of (3) remain, with the signs inherited from (2), and hence from (1). These statements indicate the sense in which $S_3, S_5, S_7$ are the members of the $\Sigma_3$ family with ancestor $S_3$, and in which $S_5$ is the child of its $S_3$ parent, and $S_7$ is the grandchild of $S_3$. The continuation of the family to realise further descendents follows a clear and unique path. As noted, (4) is not a descendent of (3), but is the parent of (26).

Some further comments address tne complexity of the results of the $S_{11}$ or $S_{13}$ members of the family $\Sigma_3$. The first two terms of (2), $\frac{1}{1} + \frac{1}{5}$ contain the integers of $G_5$. Then the denominator $(1 - x^6) = (1 - x^{P_5})$ of $f_5(x)$ and integration , generates the rest of the (2)

$$(1 + \frac{1}{5}) - (\frac{1}{7} + \frac{1}{11}) + (\frac{1}{11} + \frac{1}{13}) - \ldots \tag{12}$$

Similarly, the first bracketed set of $g_8$ terms of (3) contain the integers of $G_8$, and the denominator $(1 - x^{30}) = (1 - x^{P_7})$ of $f_7(x)$, and integration, generates the rest of (3). For the $k = 11$ member of the $\Sigma_3$ family, $g_{11} = 48$ and the denominator function is $(1 - x^{210}) = (1 - x^{P_{11}})$. This case is addressed in full below.

# 4 Evaluation of integrals

Turn next to the evaluation of integrals in (1 – 4), and thereby to completion of the proofs of these four Pi Formulas. The integrals in (1) and (2) can be evaluated by the standard procedures of the integral calculus. But (3) cannot be treated in this way. However conversion of the integrals involved in all four cases into contour integrals around a closed semicircular contour $C$ in the upper half complex plane allows residue theory to be employed. This offers a direct path to verifying the truth of (2) and (3) and others displayed below. This is illustrated now for (2).

To obtain a complex integral of the type required for (2), note that

$$\int_0^1 \frac{1+x^4}{1+x^6}dx = \int_1^\infty \frac{1+x^4}{1+x^6}dx = \frac{1}{4}\int_{-\infty}^\infty \frac{1+x^4}{1+x^6}dx \tag{13}$$

follows use of a change $y = 1/x$ of variables for the first step. This leads to the complex integral

$$\oint_C \frac{1+z^4}{1+z^6}dz = \oint_C \frac{g(z)}{h(z)}dz \tag{14}$$

where $h(z) = 1 + z^6$ has simple poles within $C$ for $z = \exp(\frac{i\pi}{6}), \exp(\frac{i\pi}{2}), \exp(\frac{5i\pi}{6})$. If $z = e^{i\theta}$ is of any one of these, its residue is given [11] by

$$\frac{g(z)}{h'(z)} = \frac{2\cos(2\theta)}{6z^3}, \tag{15}$$

and the correctness of (2) is confirmed, much more easily than by more elementary methods.

Now turn to (3), using

$$J = \oint_C \frac{(1-z^6)(1-z^{10})(1+z^{12})}{(1+z^{30})}dz. \tag{16}$$

For $J$ there are 15 poles inside the contour $C$ for

$$z_r = \exp(\frac{(2r-1)i\pi}{30}), \quad r = 1, 2, \ldots, 15. \tag{17}$$

If $z = e^{i\theta}$ is any one of these, its residue is

$$\frac{-8\sin(3\theta)\sin(5\theta)\cos(6\theta)}{30z^{15}} \tag{18}$$

Also $z^{15} = \pm i$ for all these poles. Proof that (3) is correct can now be completed.

The fourth Pi Formula in the $\Sigma_3$ family of $(1-4)$ is considered next. For the series on its left hand side the $g_{11} = 48$ terms of its first bracketed set of terms follows from the series of (3) by excising all fractions with a denominator divisible by 7, and extended to cover all of $S_{11}$ by use of the denominator $(1 + x^{P_{11}}) = (1 + x^{210})$ and integration. The result can be expressed in the form

$$K = \int_0^1 f_{11}(x)dx = \int_0^1 \frac{\psi_{11}(x)}{1+x^{210}}dx \tag{19}$$

where $\psi_{11}(x)$ is a polynomial degree 208 with 48 terms. It seems to be futile to try to use contour integration to evaluate $K$, since (a) no factorisation or other simplification of $\psi_{11}(x)$ has been found, and (b) $C$ has 105 poles within it. A new approach is called for.

A suitable approach emerges as follows. Begin by stating some results that are easy to verify. First

$$f_5(x) = f_3(x) + x^2 f_3(x^3), \tag{20}$$

which gives, upon using the change $y = x^3$ to treat the integral of the second term, the result

$$\int_0^1 f_5(x)dx = \int_0^1 f_3(x)dx(1 + \frac{1}{3}) = \frac{\pi}{3}, \tag{21}$$

in agreement with (2). Second

$$f_7(x) = f_5(x) - x^4 f_5(x^5), \tag{22}$$

gives, upon using the change $y = x^5$ to treat the integral of the second term,

$$\int_0^1 f_7(x)dx = \int_0^1 f_5(x)dx(1 - \frac{1}{5}) = \frac{4\pi}{15}, \tag{23}$$

in agreement with (3). This prompts consideration of

$$f_{11}(x) = f_7(x) + x^6 f_7(x^7). \tag{24}$$

For the left side of (24), see (19). Since $f_7(x)$ is known, the right hand side of (24) can be calculated using (22). Agreement with the findings for the left hand side has been shown to hold, certainly as far as the first 96 terms, that is, the first two bracketed sets of 48 terms, are concerned. Hence, for the fourth or $S_{11}$ member of the $\Sigma_3$ family, the result

$$\int_0^1 f_{11}(x)dx = \int_0^1 f_7(x)dx(1+\frac{1}{7}) = \frac{32\pi}{105} \tag{25}$$

is advanced.

The approach just developed also gives rise to another result, the $S_7$ descendent Pi Formula of (4). It leads directly to

$$\begin{aligned}&(1+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\frac{1}{23}-\frac{1}{29})+\\&(\frac{1}{31}+\frac{1}{37}-\frac{1}{41}+\frac{1}{43}-\frac{1}{47}+\frac{1}{49}-\frac{1}{53}-\frac{1}{59})+\dots\\&=\int_0^1 g_7(x)dx.\end{aligned} \tag{26}$$

where

$$g_7(x) = \frac{(1+x^6)(1-x^{10})(1+x^{12})}{(1-x^{30})}. \tag{27}$$

Hence, reading $g_5(x) = (1-x^4)/(1-x^6)$ off (4),

$$g_7(x) = g_g(x) + x^4 g_5(x^5) \tag{28}$$

follows, and the series of (26) sums to

$$\int_0^1 g_5(x)dx(1+\frac{1}{5}) = \frac{\sqrt{3}\pi}{5}. \tag{29}$$

Contour integration confirms this.

There is no practical obstacle to treating the $S_{11}$ grandchild of (4).

A further point to note. Since (7) deals with sets of positive integers, the sign of its second term is always negative. Pi Formulas, on the other hand, contain series with well-defined patterns of signs, which, in formulas such as (20), (22) and (28) require a case by case check on the correct sign to use for their second terms.

# 5 More Pi Formulas

Before turning attention to the discovery of Pi Formulas beyond the first few or perhaps only the first member of any family of such formulas using the fast algebraic approach, it should be noted that elementary methods or contour integation provide the only way to get any series started.

For the discussion of $S_7$ results to follow, some suitable notation is called for to avoid writing out results like those of (3) and (26) in their displayed extended forms. All the results to be stated involve the same sets of eight fractions in their successive bracketed sets. But the distribution of the signs differ from case to case. Clearly the series of (3) and (26) are fully specified by the sign patterns of their fractions via

$$\begin{aligned}&(+--++--+)-\\&(++-+-+--)+\end{aligned} \tag{30}$$

where the right most signs indicate results involving $(1 \pm x_{30}) = (1 \pm x_{P_7})$ respectively.

This enables a reasonably economical presentation of the following set of eight $S_7$ results

$$(++++++++)- \qquad \frac{(1+x^6)(1+x^{10})(1+x^{12})}{1+x^{30})} \qquad \frac{4\sqrt{3}\pi}{15}\cos(\pi/10) \tag{31}$$

$$(+--++--+)- \qquad \frac{(1+x^6)(1-x^{10})(1-x^{12})}{1+x^{30})} \qquad \frac{4\pi}{15} \tag{32}$$

$$(+-+--+-+)- \qquad \frac{(1-x^6)(1+x^{10})(1-x^{12})}{(1+x^{30})} \qquad \frac{4\sqrt{3}\pi}{15}\sin(\pi/5) \tag{33}$$

$$(++----++)- \qquad \frac{(1+x^6)(1-x^{10})(1-x^{12})}{(1+x^{30})} \qquad \frac{2\sqrt{5}\pi}{15} \tag{34}$$

$$(+---+++-)+ \qquad \frac{(1+x^6)(1-x^{10})(1+x^{12})}{(1-x^{30})} \qquad \frac{\pi}{\sqrt{15}} \tag{35}$$

$$(++-+-+--)+ \qquad \frac{(1+x^6)(1-x^{10})(1+x^{12})}{(1-x^{30})} \qquad \frac{\sqrt{3}\pi}{5} \tag{36}$$

$$(+-++-+--)+ \qquad \frac{(1-x^6)(1-x^{10})(1+x^{12})}{(1-x^{30})} \qquad \frac{\pi}{15}\sqrt{25-2\sqrt{5}} \tag{37}$$

$$(+++-+---)+ \qquad \frac{(1+x^6)(1+x^{10})(1-x^{12})}{(1-x^{30})} \qquad \frac{\pi}{15}\sqrt{25+2\sqrt{5}} \tag{38}$$

The sums to the rightmost end of each line are those of the series specified by the sign patterns at the leftmost end, using the integrands shown. All of these results have been obtained by contour integration. Eq. (32) is identical to (3), and Eq. (36) to the result (29) obtained above for the $S_7$ child of (4).

Informallly it might be suggested that some of the results above merit the label smooth. In order of smoothness therefore perhaps place (3) first, and follow with (35), (36) and (34).

In all the examples displayed on the $S_7$ scene so far, all the integrands used contain an even number of minus signs, else the first step of the analogue of (13) fails. Eq. (5), for example, can be evaluated by elementary means but not by contour integration. Eq. (5), however, is the first member of a family of series, not of course providing Pi Formulas, in which the method used above to pass to further members of the family can nevertheless be employed. It yields an $S_7$ series with sign pattern

$$(+-++--+-)- \tag{39}$$

and integral

$$\int_0^1 h_7(x)dx, \quad h_7(x) = \frac{(1-x^6)(1+x^{10})(1+x^{12})}{(1+x^{30})}, \tag{40}$$

which cannot be evaluated using contour integration. Reading the integrand $h_5(x) = (1-x^4)/(1+x^6)$ off (5), the result

$$h_7(x) = h_5(x) + x^4 h_5(x^5), \tag{41}$$

follows, so that the $S_7$ series under consideration sums to

$$\int_0^1 h_7(x)dx = \int_0^1 h_5(x)dx(1+\frac{1}{5}) = \frac{2\sqrt{3}}{5}\log(2+\sqrt{3}). \tag{42}$$

The last paragraph provides the last of the nine $S_7$ series of known sum to hand.

# 6 Further developments

The PiFormulas given above display series defined by the simplest integrals appropriate to their $S_k$ types. Ues of more complicated ones opens avenues to a range of further PiFormulas. A few illustrations follow.

First the $S_3$ result

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{17}+\frac{1}{19}-\cdots=\int_0^1\frac{1+x^2}{1+x^4}dx=\frac{\sqrt{2}\pi}{4} \tag{43}$$

Can this smooth result possibly be new? Next the $S_5$ result

$$\begin{aligned}&(1+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{23})-\\&(\frac{1}{25}+\frac{1}{29}+\frac{1}{31}+\frac{1}{35}+\frac{1}{37}+\frac{1}{41}+\frac{1}{43}+\frac{1}{47})+\dots\\&=\int_0^1\frac{(1+x^4)(1+x^6)(1+x^{10})}{(1+x^{24})}dx=\frac{\pi}{3}(1+\sqrt{2})^{1/2}\end{aligned} \tag{44}$$

Finally the $S_5$ child of (43)

$$(1-\frac{1}{5}-\frac{1}{7}+\frac{1}{11})-(\frac{1}{13}-\frac{1}{17}-\frac{1}{19}+\frac{1}{23})+\cdots=\int_0^1\frac{(1-x^4)(1-x^6)}{(1+x^{12})}dx=\frac{\sqrt{2}\pi}{6}, \tag{45}$$

resembling, but of course distinct from (5).

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