BUSSTEPP 2026
Advanced Quantum Field Theory

Peter Millington
peter.millington@manchester.ac.uk

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Advanced versus contemporary versus complicated

We often assume that lectures on "advanced" material are lectures on "contemporary" material, but the latter need not be "advanced", and neither needs to be "complicated". In quantum field theory, there is a great deal of "advanced" material that is far from modern, and much is far from complicated: "advanced", at least in *the World according to Pete*, refers to material that builds on something that has been seen before, and this, of course, is a relative and subjective statement.

The aim then is to cover aspects of quantum field theory that you likely have not been exposed to at undergraduate level and hopefully to cover aspects that will give you a basis for a range of applications of this powerful framework.

Exercises

The exercises are intended to be complementary to the lectures, and to give you a window on a little bit more of the quantum field theory that you might not otherwise meet. There are more than you will have time to cover in the study sessions, but, hopefully, having worked in smaller groups on individual exercise topics, you will have something interesting to feed back.

My hope is that you will take one or two things from these lectures, and those one or two things need only be a pointer to something that has interested you. You do not need to understand everything to understand something.

Conventions

Throughout these lectures, I will use the particle physicist's mostly minus signature convention ... unapologetically. All other notation should, I hope, be defined when first used.

1 Lecture 1: There is more to life than the Feynman propagator

Richard Feynman did a lot of things: some good, some bad, some cancellable. One of the bad things was to introduce a powerful diagrammatic framework for organising perturbation theory that we now know eponymously as Feynman diagrams. This turned quantum field theory into a practical tool for calculating scattering matrix elements. Great, I hear you say!? No. There is much more to quantum field theory than the successes of collider-based particle physics would lead you to believe, and this "much more" will be the focus of these lectures. I am not hating on collider physics, by the way, (or Feynman, for that matter) but perturbative matrix elements (and especially tree-level ones) need regrettably little of the full power of quantum field theory.

Of course, predicting the rate of events at a collider is a justifiable pursuit and a very cool thing to be able to do. And to Feynman's credit, we will meet some good things that he did later in this lecture series, in particular with Frank Vernon Jr. The question is: what do we want to calculate? There are all manner of things . . .

In this lecture, we are going to consider the Klein–Gordon equation for a real scalar field $\phi \equiv \phi(x)$,

$$-(\Box_x + m^2)\phi(x) = J(x) \ , \tag{1}$$

where $\Box_x = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}$ is the d'Alembertian operator and $J \equiv J(x)$ is a local source. "How inspiring!?", I hear you say. Well, let's see. The solution to Eq. (1) can be written

$$\phi(x) = \phi_0(x) + \int \mathrm{d}^4 y \ G(x, y)J(y) \ , \tag{2}$$

where $\phi_0(x)$ is the solution for $J(x) = 0$, i.e.,

$$-(\Box_x + m^2)\phi_0(x) = 0 \ , \tag{3}$$

and $G(x, y)$ is a Green's function — do have a read about George Green, the Nottinghamshire miller and mathematician — satisfying

$$-(\Box_x + m^2)G(x, y) = \delta^4(x - y) \ , \tag{4}$$

in which $\delta^4(x-y)$ is a four-dimensional Dirac delta function, really the product of four Dirac delta functions, one for each of the four coordinates.

Before we dive into technicalities, let's remind ourselves what Eq. (2) describes: the source $J(x)$ produces a disturbance in the field $\phi(x)$, which is *propagated* from the spacetime point $y \equiv y^\mu$ to the point $x \equiv x^\mu$ by $G(x, y)$.

Now, we have a dissatisfying asymmetry in x and y in our framing of the problem, but we can remedy this by realising that the Green's function must also satisfy

$$-G(x,y)(\overleftarrow{\Box}_y + m^2) = \delta^4(x-y)\ , \tag{5}$$

where the arrow indicates the direction in which the d'Alembertian acts.

We are going to proceed by moving to Fourier space, and we will make the sadistic step of transforming with respect to both x and y separately, writing

$$G(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{-ip\cdot x} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, e^{iq\cdot y}\, G(p,q)\ . \tag{6}$$

We have been somewhat judicious in our choice of relative sign in the exponents, as we will see. Thus, we need to solve

$$(p^2 - m^2)G(p,q) = 1\ , \tag{7a}$$

$$G(p,q)(q^2 - m^2) = 1\ . \tag{7b}$$

Please note that we have heavily abused notation and will differentiate the Green's function from its Fourier transform only by its argument.

To solve Eq. (7), let's start with the p dependence. We first need to understand how to deal with the case $p^2 = m^2$, which corresponds to the two *single-particle poles* at $p^0 = \pm E_{\mathbf{p}} = \pm\sqrt{\mathbf{p}^2 + m^2}$. We also need to be mindful that the Fourier transform of the Green's function is a distribution, thereby making sense only within the Fourier integral, and we have two of these. With this in mind, and after a little thought, it appears that we can add to the Green's function a term proportional to a delta function $\delta(p^2 - m^2)$ and a seemingly arbitrary function of the four-momentum p (and the other momentum q), except presumably that this must not also be singular at $p^2 = m^2$. So there does not seem to be a unique solution for the Green's function. In fact, we will obtain a family of different Green's functions, depending on how we deal with the single-particle poles, and each of these Green's functions will perform a different function. This is dealt with very nicely in Greiner and Reinhardt's *Field quantization*.

Suppose we avoid the poles altogether: we could, assuming we have nothing better to do, integrate from negative infinity to very close to the negative pole from the left ($p^0 = -E_{\mathbf{p}} - \epsilon$, $\epsilon > 0$), integrate from very close to the right of the negative pole ($p^0 = -E_{\mathbf{p}} + \epsilon$) to very close to the left of the positive pole ($p^0 = E_{\mathbf{p}} - \epsilon$), integrate from very close to the right of the positive pole ($p^0 = E_{\mathbf{p}} + \epsilon$) to positive infinity, sum the contributions, and take the limit that we get infinitely close to the two poles, i.e., $\epsilon - > 0^+$ (where the plus indicates that we approach zero from above). This is the *Cauchy principal value*. Now for some trickery: if we take two copies of this same integral, we can fill the gaps with semicircles, deformed into the complex plane that go just above the poles in one case and just below the poles in the second case. Digging into the memory banks or after a conversation with our favourite Large Language Model, we know from complex analysis that the contributions to the integral from these two semi-circles will cancel by Jordan's lemma. Taking the average, we can then express the Cauchy principal value integral as the average of two contour integrals: one, which for reasons that will become clear we call $\mathcal{C}_R$, that runs from left to right in the complex plane and skips just above the poles and one, which we'll call $\mathcal{C}_A$, that runs from left to right along a contour in the complex plane that dips just below the poles:

$$\begin{aligned} G_{\mathcal{P}}(x,y) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathcal{P} \frac{e^{-ip\cdot x + iq\cdot y}}{p^2 - m^2} g(p,q) \\ &= \frac{1}{2}\left[\int_{\mathcal{C}_R} \frac{\mathrm{d}^4 p}{(2\pi)^4} + \int_{\mathcal{C}_A} \frac{\mathrm{d}^4 p}{(2\pi)^4}\right] \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{e^{-ip\cdot x + iq\cdot y}}{p^2 - m^2} g(p,q) \;, \end{aligned} \tag{8}$$

where, by inspection, it had better be able to fix $g(p,q)$ such that

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, e^{-ip\cdot x + iq\cdot y}\, g(p,q) = 1 \;. \tag{9}$$

Rather than defining these specific contours, we can also introduce a limit representation by introducing "pole prescriptions" that capture the appropriate deformations of the contour:

$$\begin{aligned} G_{\mathcal{P}}(x,y) &= \frac{1}{2} \sum_{s=\pm 1} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{e^{-ip\cdot x + iq\cdot y}}{p^2 - m^2 + is\epsilon p^0}\, g(p,q) \\ &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, e^{-ip\cdot x + iq\cdot y}\, \frac{(p^2 - m^2)}{(p^2 - m^2)^2 + \epsilon^2}\, g(p,q) \;, \end{aligned} \tag{10}$$

where the limit $\epsilon \to 0^+$ is understood implicitly, and we have redefined $\epsilon p^0 \to \epsilon$ in the final equality, since the sign of p^0 is no longer relevant.

Had we solved instead from the right, focussing first on the q dependence, we would have obtained an analogous result with $p \leftrightarrow q$:

$$\begin{aligned} G_{\mathcal{P}}(x,y) &= \frac{1}{2} \sum_{s=\pm 1} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{e^{-ip\cdot x + iq\cdot y}}{q^2 - m^2 + is\epsilon q^0}\, g(q,p) \\ &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, e^{-ip\cdot x + iq\cdot y}\, \frac{(q^2 - m^2)}{(q^2 - m^2)^2 + \epsilon^2}\, g(q,p) \;. \end{aligned} \tag{11}$$

For both forms of the solution to hold, we conclude that

$$g(p,q) = g(q,p) = (2\pi)^4 \delta^4(p-q) \;, \tag{12}$$

and the convenience of our choice of sign convention on p and q has now delivered. We obtain a translationally invariant solution

$$G_{\mathcal{P}}(x,y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)}\,\frac{(p^2-m^2)}{(p^2-m^2)^2+\epsilon^2}\tag{13}$$

that depends only on the *relative coordinate*

$$s^\mu\equiv x^\mu-y^\mu\ .\tag{14}$$

Staring back at the first equality in Eq. (10), we have the sum of two Green's functions, with opposite pole prescriptions. The poles of the first

$$G_R(x,y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon p^0}\tag{15}$$

lie in the lower-half of the complex plane. Taking only the p^0 integral, if $x^0-y^0<0$, this integral is zero, since we are allowed to close the contour of integration in the upper-half complex plane — the curved part of the contour gives a vanishing contribution as the integrand is exponentially damped when $\mathrm{Im}\,p^0>0$ — in which case we do not enclose any poles and the integral is zero by the residue theorem. On the other hand, the second Green's function

$$G_A(x,y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2-i\epsilon p^0}\tag{16}$$

vanishes by an analogous argument if $x^0-y^0>0$. Dredging up a bit of special relativity, we recall that if two events, occurring at times x^0 and y^0 in a given frame of reference, are spacelike separated, i.e., $(x-y)^2<0$, then we can always find a frame in which the two events are simultaneous, i.e., $x'^0=y'^0$. Boosting to this frame and noting that the integration measure and the denominator of the integrand are Lorentz invariant, since the sign of p^0 is unchanged under a proper Lorentz transformation, we are then allowed to close the integration contour in whichever way we choose — upper- or lower-half complex plane — and both G_R and G_A will therefore vanish for arguments that are spacelike separated. Thus, G_R and G_A are the retarded and advanced Green's functions, which propagate disturbances in the scalar field *causally*.

Now, using the identity

$$\frac{1}{x+i\epsilon}=\mathcal{P}\frac{1}{x}-i\pi\delta(x)\ ,\tag{17}$$

we see that the retarded and advanced Green's functions differ only in the contribution from the delta function $\delta(p^2-m^2)$. More specifically, we can write

$$G_{R(A)}(p)=\mathcal{P}\frac{1}{p^2-m^2}-(+)i\pi\mathrm{sgn}(p^0)\delta(p^2-m^2)\ ,\tag{18}$$

where $\operatorname{sgn} p^0$ is the sign function. We see that G_R and G_A are two linearly independent combinations of the Cauchy principal value $G_{\mathcal{P}}$ and another function

$$G(p) = -2\pi i \operatorname{sgn}(p^0)\delta(p^2 - m^2) \ . \tag{19}$$

This is the *Pauli–Jordan function*, which is a solution to the homogeneous equation

$$(p^2 - m^2)G(p) = 0 \ . \tag{20}$$

Its inverse Fourier transform is the expectation value of the field commutator

$$iG(x, y) = \langle [\hat{\phi}(x), \hat{\phi}(y)] \rangle = iG_R(x, y) - iG_A(x, y) \ . \tag{21}$$

We use hats to denote operators for reasons that remain a mystery to me. The commutator vanishes by construction for x and y spacelike separated — this is the condition of *microcausality* that we build into the operator algebra for our scalar field theory. Since $G_R(x, y)$ vanishes for $x^0 - y^0 < 0$ and $G_A(x, y)$ vanishes for $x^0 - y^0 > 0$, we might expect that

$$iG_R(x, y) = \theta(x^0 - y^0) \langle [\hat{\phi}(x), \hat{\phi}(y)] \rangle \ , \tag{22a}$$

$$iG_A(x, y) = \theta(y^0 - x^0) \langle [\hat{\phi}(y), \hat{\phi}(x)] \rangle \ , \tag{22b}$$

where θ is the unit step function, and this is indeed the case.

We can partition the Pauli–Jordan function another way by noting that

$$\operatorname{sgn}(p^0) = \theta(+p^0) - \theta(-p^0) \ . \tag{23}$$

This allows us to introduce the positive- and negative-frequency *Wightman functions*, which we might take to be

$$iG_>(p) = 2\pi\theta(+p^0)\delta(p^2 - m^2) \ , \tag{24a}$$

$$iG_<(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2) \ , \tag{24b}$$

such that

$$G(x, y) = G_>(x, y) - G_<(x, y) \ . \tag{25}$$

They, too, are solutions to the homogeneous equation (20), and their sum defines the *Hadamard function*

$$iG_1(k) = iG_>(k) + iG_<(k) = 2\pi\delta(p^2 - m^2) \ . \tag{26}$$

This is nothing other than the expectation value of the field anticommutator

$$iG_1(x,y) = \langle \{ \hat{\phi}(x), \hat{\phi}(y) \} \rangle \ , \tag{27}$$

and the Wightman functions are the ordered expectation values

$$iG_>(x,y) = \langle \hat{\phi}(x)\hat{\phi}(y) \rangle \ , \tag{28a}$$

$$iG_<(x,y) = \langle \hat{\phi}(y)\hat{\phi}(x) \rangle \ . \tag{28b}$$

Looking at the various pole prescriptions, we have four possibilities, corresponding to placing poles at $p^0 = \pm E_{\mathbf{p}} \pm' i\epsilon$. Taking both poles in the lower-half plane gives the retarded Green's function, taking both in the upper-half plane gives the advanced Green's function, but we can also take poles in diagonally opposite quadrants. Doing so gives the time- and antitime-ordered Green's functions

$$iG_{\mathrm{F}}(x,y) = \langle \mathrm{T}\big[\hat{\phi}(x)\hat{\phi}(y)\big] \rangle = \theta(x^0 - y^0) iG_>(x,y) + \theta(y^0 - x^0) iG_<(x,y) \ , \tag{29a}$$

$$iG_{\mathrm{D}}(x,y) = \langle \bar{\mathrm{T}}\big[\hat{\phi}(x)\hat{\phi}(y)\big] \rangle = \theta(x^0 - y^0) iG_<(x,y) + \theta(y^0 - x^0) iG_>(x,y) \ . \tag{29b}$$

The Fourier transform of the first of these coincides with the Feynman propagator, the second with the Dyson propagator — Dyson deserves some credit, too!

So, are we done? No. We noticed earlier that we can seemingly add as much of the solution to the homogeneous equation as we want to the Green's function. So how much should we add?

First, let's look again at the Pauli–Jordan function. When we partitioned it into the Wightman functions, we missed that we can include additional delta function terms, so long as they cancel when we take the difference, i.e.,

$$iG_>(p,q) = 2\pi \big[\theta(+p^0) g(p,q) + h(p,q)\big] \delta(p^2 - m^2) \ , \tag{30a}$$

$$iG_<(p,q) = 2\pi \big[\theta(-p^0) g(p,q) + h(p,q)\big] \delta(p^2 - m^2) \ . \tag{30b}$$

But we need the term proportional to $h(p,q)$ also to solve the left-acting homogeneous equation

$$G_{\gtrless}(p,q)(q^2 - m^2) = 0 \ . \tag{31}$$

Thus, $h(p,q)$ must also be proportional to $\delta(q^2 - m^2)$, and we write

$$h(p,q) = f(p,q)\delta(q^2 - m^2) \tag{32}$$

The most general form of the Hadamard function is then

$$iG_1(k) = 2\pi\delta(p^2-m^2)\big\{\big[\theta(+p^0)\theta(+q^0)+\theta(-p^0)\theta(-q^0)\big](2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})+2f(p,q)\big]2\pi\delta(q^2-m^2)\ . \tag{33}$$

With the inverse Fourier transform in mind, we see that this corresponds to a momentum q flowing out of the point y and a momentum p flowing into the point x, and these needn't be the same, in general. Why? We'll come to this.

By virtue of their definitions, we can show that

$$G_{\rm F}(p,q) = G_R(p,q) + G_<(p,q) = G_A(p,q) + G_>(p,q)\ , \tag{34a}$$

$$G_{\rm D}(p,q) = -G_R(p,q) + G_>(p,q) = -G_A(p,q) + G_<(p,q)\ , \tag{34b}$$

such that

$$iG_{\rm F}(p,q) = +i\frac{(2\pi)^4\delta^4(p-q)}{p^2-m^2+i\epsilon} + 2\pi\delta(p^2-m^2)f(p,q)2\pi\delta(q^2-m^2)\ , \tag{35a}$$

$$iG_{\rm D}(p,q) = -i\frac{(2\pi)^4\delta^4(p-q)}{p^2-m^2-i\epsilon} + 2\pi\delta(p^2-m^2)f(p,q)2\pi\delta(q^2-m^2)\ . \tag{35b}$$

What then is it that fixes the function $f(p,q)$?

Up to this point, we have not specified the expectation value of the field operators with which we are identifying these Green's functions and their components. The most general expectation value will be one that allows for the most general state of the system. This will be described by some density operator $\hat{\rho}$. If we work in the Heisenberg picture, where the operators carry all of the time dependence, and the states do not evolve in time then, for an isolated system, $\hat{\rho}$ is time independent, and the expectation value of some operator $\hat{O}(x,y,\dots)$

$$\langle\hat{O}(x,y,\dots)\rangle = \frac{1}{Z}\,{\rm tr}\,\hat{\rho}\,\hat{O}(x,y,\dots)\ , \tag{36}$$

where $Z = {\rm tr}\,\hat{\rho}$ is the partition function. The latter will be unity if the density operator is properly normalised. We'll come back to the partition function in the next lecture.

Let's start with the Pauli–Jordan function, it is the expectation value of the commutator, but the commutator of the field operators is proportional to the identity operator, and therefore the form of the Pauli–Jordan function is independent of the state of the system encoded in $\hat{\rho}$. Since the retarded and advanced Green's functions are proportional to the expectation value of the commutator, they too will be independent of the state, as will the Cauchy principal value $G_{\mathcal{P}}$ constructed from them. On the other hand, the Wightman and Hadamard functions will depend on the state of the system, and this will be captured by the unknown function $f(p,q)$. This makes sense: the density operator $\hat{\rho}$ contains information about whatever ensemble of scalar particles our system is made up of, and so the state modifies the part of the Green's function that depends on the single-particle poles that correspond to these particles.

The state need not be spacetime translationally invariant, in which case there is no reason for four-momentum to be conserved through the propagator, and $p \neq q$. Moreover, if we were to promote our simple scalar field to something more complicated with a set κ or λ of quantum numbers then, in general, the function $f(p,q)$ will be promoted to $f_{\kappa\lambda}(p,q)$. In this case, we would write the Feynman propagator in the form

$$iG_{\rm F}(p,q) = i\frac{(2\pi)^4\delta_{\kappa\lambda}\delta^4(p-q)}{p^2-m_\kappa^2+i\epsilon} + 2\pi\delta(p^2-m_\kappa^2)f_{\kappa\lambda}(p,q)2\pi\delta(q^2-m_\lambda^2)\ . \tag{37}$$

The function $f_{\kappa\lambda}(p,q)$ can then encode any off-diagonal correlations between the quantum numbers κ and λ that define the state of the system. We can do the same for higher spin fields, with manifold increases in technical complication, but our simple scalar playground is enough for a Monday.

Returning then, we have $\hat{\rho} = |0\rangle\langle 0|$ in vacuum, and we might expect that $f(p,q) = 0$. We can readily confirm this by making use of the usual canonical algebra for the scalar field, expressed in terms of creation and annihilation operators $\hat{a}^\dagger(\mathbf{k})$ and $a(\mathbf{k})$, and calculating the expectation values explicitly. In this case, the Feynman and Dyson propagators take their familiar forms

$$iG_{\rm F}(p) = \frac{+i}{p^2-m^2+i\epsilon}\ , \tag{38a}$$

$$iG_{\rm D}(p) = \frac{-i}{p^2-m^2-i\epsilon}\ , \tag{38b}$$

wherein we have integrated over the redundant additional momentum q. The Fock vacuum is Poincaré-invariant, and we have obtained a translationally invariant Green's function built only out of the Lorentz scalar p^2 and the pole mass m^2. In the next lecture, we will go beyond the case of the Fock vacuum $|0\rangle$.

Signposting from Lecture 1

In this lecture, we have:

- obtained the general form of the Green's and related correlation functions for a real scalar field in Minkowski spacetime, allowing for a non-trivial macroscopic state that need not be spacetime translationally invariant.

In the next lecture, we will:

- consider a thermal state, derive the corresponding form of the as-yet-unknown function $f(p,q)$, and consider two alternative treatments: the Matsubara or imaginary-time formalism and thermo field dynamics.