BUSSTEPP 2026
Advanced Quantum Field Theory
Exercise Sets

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The exercises are intended to be complementary to the lectures, and to give you a window on a little bit more of the quantum field theory that you might not otherwise meet. There are more than you will have time to cover in the study sessions, but, hopefully, having worked in smaller groups on individual exercise topics, you will have something interesting to feed back.

My hope is that you will take one or two things from these lectures, and those one or two things need only be a pointer to something that has interested you. You do not need to understand everything to understand something.

Exercises

Exercise 1: Wigner functions

In the first lecture, we work in an enlarged momentum space. On many occasions, however, it turns out to be convenient to instead work in a space spanned by one momentum and one coordinate. This approach is often named for Eugene Wigner.

The Wigner functions are defined as the inverse Fourier transforms of our two-momentum Green's functions with respect to the *relative momentum* $Q^\mu = p^\mu - q^\mu$, which is conjugate to the *central coordinate*

$$X^\mu = \frac{x^\mu + y^\mu}{2} \ .$$

This yields functions that depend on X and the central momentum

$$k^\mu = \frac{p^\mu + q^\mu}{2} \ ;$$

for example

$$G(k, X) = \int \frac{\mathrm{d}^4 Q}{(2\pi)^4} \, e^{-iQ.X} \, G(k + Q/2, k - Q/2) \ .$$

- Assuming spatial translational invariance, derive a general expression for the Wigner space representation of the Feynman and Dyson propagators by transforming the expressions obtained in the lecture.

- By inserting the Wigner transform, show that the convolution of two functions of the coordinates x, y and z

  $$\int \mathrm{d}^4 z \ A(x, z) B(z, y) \ ,$$

  can be expressed in Wigner space as

  $$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \, e^{-i\diamond} \{A(k, X)\}\{B(k, X)\} \ ,$$

  where the diamond operator is expressed in terms of the *Poisson bracket* via

  $$\diamond\{A\}\{B\} = \frac{1}{2}\{A, B\}_{k,X} = \frac{1}{2}\left[\frac{\partial A}{\partial k^\mu}\frac{\partial B}{\partial X_\mu} - \frac{\partial A}{\partial X_\mu}\frac{\partial B}{\partial k^\mu}\right] \ .$$

  This is related to the *Moyal bracket*, named for José Enrique Moyal but also independently defined by Hip Groenewold; look into this if you have time.

Exercise 2: Breakdown of perturbation theory at finite temperature

Parts of this calculation are nicely detailed in the Cambridge monograph *Thermal field theory* by Michel Le Bellac. Starting with the path-integral representation of the partition function for the massless ϕ^4 theory

$$Z(\beta) = \int \mathcal{D}\phi \ \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \ \left(-\frac{1}{2!}\,\phi\partial^2\phi + \frac{\lambda}{4!}\phi^4\right)\right] \ :$$

- Show that the temperature-dependent part of the one-loop tadpole diagram is

$$\frac{\lambda T}{2}\sum_n \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_n^2+\omega_{\mathbf{k}}^2} \supset \frac{\lambda T^2}{24} \ .$$

  You might need to look up the infinite summation.

- Hence, derive the order λ quantum correction to the entropy of a gas of scalar particles. Hint: Expand the exponent in the partition function to first order in λ, use Wick's theorem, and insert the result above for the tadpole, as appropriate.

- Resumming the tadpole into the Matsubara propagator gives

$$G^{-1}(\omega_n,\mathbf{k}) = \omega_n^2 + \omega_{\mathbf{k}}^2 + \frac{\lambda T^2}{24} \ .$$

  Hence, or otherwise, show that the resummation of one-loop tadpole insertions provides a correction to the entropy of the form

$$S \supset \frac{\pi^2 V T^3}{12}\left(\frac{\lambda}{24\pi^2}\right)^{3/2} \ .$$

  Hint: You will need to subtract off the order λ term that you calculated above in order to obtain a finite result. This is known as the *ring* or *daisy* resummation. Notice that the result is not analytic in the coupling λ, and this breakdown in perturbation theory is due to infrared divergences. Finite temperature, and worse being out of thermodynamic equilibrium, unleashes all sorts of carnage on perturbation theory.

Exercise 3: Unitarity and the Schwinger–Keldysh path integral

By considering the Schwinger–Keldysh path integral for the massive ϕ^4 theory:

- Working at order λ, show that the vacuum graphs cancel. This is actually a consequence of unitarity, and the *largest time equation* — have a look into this — which is encoded in the Schwinger–Keldysh path integral by virtue of its construction.

- Rewrite the Schwinger–Keldysh path integral

$$Z = N \int \mathcal{D}\pi_i^+ \int \mathcal{D}\pi_i^- \; \delta(\pi_i^+ - \pi_i^-) \\ \times \int \mathcal{D}\phi_i^+ \int \mathcal{D}\phi_i^- \int \mathcal{D}\phi^+ \int \mathcal{D}\phi^- \; \langle \phi^+(t_i)|\hat{\rho}(t_i)|\phi^-(t_i)\rangle \; e^{iS[\phi^+]-iS[\phi^-]} \; ,$$

  in terms of the following variables:

$$\phi^c = \frac{\phi_+ + \phi_-}{2} \qquad \text{and} \qquad \phi^q = \phi_+ - \phi_- \; .$$

  N is some normalisation that we are never particularly careful about, and we have isolated the functional integrals over the field variables at time t_i.

- Re-express the path integral in terms of a Wigner *functional*. After dropping terms higher than first order in ϕ_q, show that the functional integration over ϕ_q imposes the classical equation of motion on ϕ_c. This is known as the *classical statistical approximation*, and it allows us to sample from an initial distribution — here the Wigner functional — and evolve it according to the classical equations of motion.

Exercise 4: Back to scattering matrix elements

Throughout the lectures, I emphasise the calculation of expectation values. In what follows, we will see that we can obtain scattering probabilities from expectation values.

The probability of a given measurement outcome $\mathbb{P}$, given a density operator $\hat{\rho}(t)$, can be written in terms of the expectation value of an *effect operator E* in the following form:

$$\mathbb{P} = \mathrm{tr}\,\hat{\rho}(t)E \; .$$

- Making use of the unitary time evolution operator convince yourself that we recover an expression involving the usual squared matrix element for a pure initial state, described by the density operator $\hat{\rho}(t_i) = |i\rangle\langle i|$ and a pure final state projection operator $\hat{E} = |f\rangle\langle f|$.

By making use of the the Baker–Campbell–Hausdorff formula, we can obtain an expression for the probability in terms of nested commutators, which takes the form

$$\mathbb{P} = \sum_{j=0}^{\infty} \int_{t_i}^{t_f} \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_j \; \Theta_{12\ldots j} \, \mathrm{tr}\,\hat{\rho}_i \, \hat{\mathcal{F}}_j \; ,$$

where $\Theta_{12\ldots j}$ imposes that the times are ordered with $t_f > t_1 > t_2 > \cdots > t_j > t_i$, $\hat{\rho}_i$ is the initial state density operator, $\hat{\mathcal{F}}_0 = E$, and

$$\hat{\mathcal{F}}_j = -i\big[\hat{\mathcal{F}}_{j-1}, \hat{H}^{\rm int}_j\big] \ ,$$

expressed in terms of the interaction part of the Hamiltonian $\hat{H}^{\rm int}$ at time t_j.

- Show that the probability is unity for a fully inclusive final state. Hint: Think about what the effect operator that corresponds to "any state" should be.
- Taking as an example the $\lambda\phi\chi^2$ theory, where ϕ and χ are real scalar fields, calculate the probability for two χ's to zero χ's for any number of final state ϕ's at order λ^2. You should assume that there are no initial state ϕ's. Which process does this correspond to? Hint: Take the final state effect operator to be $\hat{E}(t) = \hat{\mathbb{I}}^\phi \otimes |0^\chi\rangle\,\langle 0^\chi|$.

If you are interested to read more about this approach, see the works by subsets of Robert Dickinson, Jeff Forshaw, Ross Jenkinson and myself.

Exercise 5: A pseudo-Hermitian 2×2 matrix problem

Consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \qquad a, b, c, d \in \mathbb{R} \ .$$

- Solve for the eigenvalues and eigenvectors of the matrix $\boldsymbol{A}$ for the case $c = -b$. When are the eigenvalues real? When are they complex? What happens to the eigenvalues and eigenvectors of $\boldsymbol{A}$ when $b^2 = (a-d)^2/4$?
- Show that the eigenvectors $\mathbf{e}_i$ are not orthogonal with respect to the inner product $\mathbf{e}_i^* \cdot \mathbf{e}_j$.
- Show that there exist two matrices

  $$\boldsymbol{\eta}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\eta}_2 = \frac{1}{\sqrt{1-\xi^2}}\begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix} \ , \qquad \xi = \frac{2b}{a-d} \ ,$$

  that serve as metrics for inner products of the form $\mathbf{e}_i^* \cdot \boldsymbol{\eta} \cdot \mathbf{e}_j$, with respect to which the eigenvectors are orthogonal. Which of the two inner products yields positive norms for both eigenvectors?
- Show that $\boldsymbol{A}^\dagger = \boldsymbol{\eta}_{1(2)} \cdot \boldsymbol{A} \cdot \boldsymbol{\eta}_{1(2)}^{-1}$ for both $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$.

You are welcome to do this by hand, but I would recommend using your favourite common-or-garden symbolic algebra package.