Structure and Evolution of Stars

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[For Examples class 1.30-3.30pm Tuesday 24th October] Michaelmas Term, 2023

Examples Sheet I

1. Sirius A has an apparent visual magnitude $m_{\rm V} = -1.5$ and a spectral type A0V. Its distance is 2.7 pc. Calculate its luminosity and radius in solar units.

Sirius B is a binary companion to Sirius A. It has the same spectral type and $m_{\rm V} = 8.5$. Calculate its luminosity and radius.

Measurements of the orbits of A and B give a binary period of 50 years, mass ratio $M_{\rm A}/M_{\rm B} = 3$ and a semi major-axis of 8 seconds of arc.

Calculate the masses and mean densities of both components. Which star is the main-sequence star and what is the other?

[*For an* A0V *spectrum you may assume* $T_{\rm eff} = 10{,}000\,{\rm K}$ *and Bolometric Correction,* $m_{\rm bol} - m_{\rm V} = -0.4$. *An absolute bolometric magnitude* $M_{\rm bol}$ *of zero corresponds to a luminosity of* $3.0 \times 10^{35}\,{\rm erg\,s^{-1}}$. *For a binary orbit you may use the fact that the semimajor axis a, total mass M and period P are related by Kepler's law,*

$$\left(\frac{a}{\rm au}\right)^3 = \frac{M}{M_\odot}\left(\frac{P}{\rm years}\right)^2 .]$$

2. In any equilibrium configuration prove that

$$\frac{d}{dr}\left(P + \frac{Gm^2}{8\pi r^4}\right) < 0,$$

where $m(r) = \int_0^r 4\pi r^2 \rho dr$. Deduce a lower limit for the central pressure $P_{\rm c}$ in terms of the stellar mass and radius. Evaluate the limit for the Sun.

3. Show that the gravitational energy of a spherically symmetric star is

$$\Omega = -4\pi \int_0^R \rho \frac{Gm}{r} r^2 dr.$$

Hence show that a polytrope of index $n$ has

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R}.$$

The recombination energy of stellar matter is $I$ per unit mass. Show that if the internal structure of a star corresponds to an $n = \frac{3}{2}$ polytrope total disruption of the star is energetically feasible if

$$R > \frac{3}{7}\frac{GM}{I}.$$

Given that the ionization energy of a hydrogen atom is 13.6 eV estimate the maximum radius for a star of $1\,M_\odot$ and explain the existence of red giants.

4. By using the virial theorem, or otherwise, show that the period $\Pi$ of spherically symmetric pulsations of a star satisfying $I = 4\pi\int_0^R r^4\rho dr = kMR^2$ and $\Omega = -\eta GM^2/R$, where $k$ and $\eta$ are constants, is given by

$$\Pi = \left[\frac{3\pi k}{(3\gamma - 4)\eta G\bar{\rho}}\right]^{\frac{1}{2}},$$

where $\bar{\rho}$ is the mean density of the star.

Comment on the relation of $\Pi$ to the free-fall time and the acoustic travel time from surface to centre.

5. If $\rho$ decreases outwards, show that $m < \frac{4}{3}\pi r^3\rho_{\mathrm{c}}$.

Prove that $P_{\mathrm{c}} < \frac{1}{2}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}\rho_{\mathrm{c}}^{\frac{4}{3}}M^{\frac{2}{3}}$.

Deduce that if $\beta = P_{\mathrm{gas}}/P$, then

$$1 - \beta_{\mathrm{c}} \leq 1 - \beta^*,$$

where $beta^*$ satisfies Eddington's quartic,

$$M = \left(\frac{6}{\pi}\right)^{\frac{1}{2}}\left[\left(\frac{R}{\mu_{\mathrm{c}}}\right)^4\frac{3}{a}\left(\frac{1-\beta^*}{\beta^{*4}}\right)\right]^{\frac{1}{2}}G^{-\frac{3}{2}}.$$

Evaluate $M$ for $1 - \beta^* = 0.2, 0.5, 0.8$ and comment on what you find.