1. Sirius A has an apparent visual magnitude $m_V = -1.5$ and a spectral type A0V. Its distance is 2.7 pc. Calculate its luminosity and radius in solar units.

Sirius B is a binary companion to Sirius A. It has the same spectral type and $m_V = 8.5$. Calculate its luminosity and radius.

Measurements of the orbits of A and B give a binary period of 50 years, mass ratio $M_A/M_B = 3$ and a semi major-axis of 8 seconds of arc.

Calculate the masses and mean densities of both components. Which star is the main-sequence star and what is the other?

[For an A0V spectrum you may assume $T_{\text{eff}} = 10,000$ K and Bolometric Correction, $m_{\text{bol}} - m_V = -0.4$. An absolute bolometric magnitude $M_{\text{bol}}$ of zero corresponds to a luminosity of $3 \times 10^{35}$ erg s$^{-1}$. For a binary orbit you may use the fact that the semimajor axis $a$, total mass $M$ and period $P$ are related by Kepler’s law,

$$
\left( \frac{a}{\text{au}} \right)^3 = \frac{M}{M_\odot} \left( \frac{P}{\text{years}} \right)^2.
$$

2. In any equilibrium configuration prove that

$$
\frac{d}{dr} \left( P + \frac{Gm^2}{8\pi r^4} \right) < 0,
$$

where $m(r) = \int_0^r 4\pi r^2 \rho dr$. Deduce a lower limit for the central pressure $P_c$ in terms of the stellar mass and radius. Evaluate the limit for the Sun.

3. Show that the gravitational energy of a spherically symmetric star is

$$
\Omega = -4\pi \int_0^R \rho \frac{Gm}{r} r^2 dr.
$$

Hence show that a polytrope of index $n$ has

$$
\Omega = -\frac{3}{5-n} \frac{GM^2}{R}.
$$
The recombination energy of stellar matter is $I$ per unit mass. Show that if the internal structure of a star corresponds to an $n = \frac{3}{2}$ polytrope total disruption of the star is energetically feasible if

$$R > \frac{3 GM}{I}.$$ 

Given that the ionization energy of a hydrogen atom is 13.6 eV estimate the maximum radius for a star of $1 M_\odot$ and explain the existence of red giants.

4. By using the virial theorem, or otherwise, show that the period $\Pi$ of spherically symmetric pulsations of a star satisfying $I = 4\pi \int_0^R r^4 \rho dr = kMR^2$ and $\Omega = -\eta GM^2/R$, where $k$ and $\eta$ are constants, is given by

$$\Pi = \left[ \frac{3\pi k}{(3\gamma - 4)\eta G \bar{\rho}} \right]^\frac{1}{2},$$

where $\bar{\rho}$ is the mean density of the star.

Comment on the relation of $\Pi$ to the free-fall time and the acoustic travel time from surface to centre.

5. If $\rho$ decreases outwards, show that $m < \frac{4}{3} \pi r^3 \rho_c$.

Prove that $P_c < \frac{1}{2} G \left( \frac{4}{3} \pi \right)^{\frac{1}{3}} \rho_c^\frac{4}{3} M^\frac{2}{3}$.

Deduce that if $\beta = P_{\text{gas}}/P$, then

$$1 - \beta_c \leq 1 - \beta^*,$$

where $\beta^*$ satisfies Eddington’s quartic,

$$M = \left( \frac{6}{\pi} \right)^{\frac{1}{2}} \left[ \left( \frac{R}{\mu c} \right)^4 \frac{3}{a} \left( \frac{1 - \beta^*}{\beta^*} \right) \right]^\frac{1}{2} G^{-\frac{3}{2}}.$$

Evaluate $M$ for $1 - \beta^* = 0.2, 0.5, 0.8$ and comment on what you find.

6. Take pure, partially ionized hydrogen gas, assume $P = nkT$ thus neglect radiation pressure. Then $n = n_H + n_e$. Let $x = \frac{n_e}{n_H}$ so $x$ is a measure of the fractional ionization of hydrogen.

Calculate $\Gamma_1$ in terms of $x$ and $R = \frac{\chi}{kT}$ where $\chi$ is ionization potential of hydrogen. Discuss the dependence of $\Gamma_1$ on $x$. 

2