Structure and Evolution of Stars

C. A. Tout

[For Examples class 1.30-3.30pm Tuesday 12th November] Michaelmas Term, 2024

Examples Sheet II

1. An enclosure containing a perfect gas (specific heat ratio γ) and radiation undergoes an adiabatic change. Show that

$$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0,$$

$$\text{where} \quad \Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta + 12(\gamma-1)(1-\beta)}.$$

Sketch how Γ_1 varies for $0 < \beta \leq 1$.

2. By use of the Fermi–Dirac distribution for electrons

$$N_p \mathrm{d}p = \frac{8\pi}{h^3} \frac{p^2 \mathrm{d}p}{\exp\left[(p^2/2mkT) - \Psi\right] + 1},$$

where Ψ is defined in terms of the number density N by

$$N = \int_0^\infty N_p \mathrm{d}p,$$

show that the electron pressure P_e and density ρ can be written as

$$P_\mathrm{e} = \frac{4\pi}{h^3} (2mkT)^{3/2}\, kT \left[\frac{2}{3} F_{3/2}(\Psi)\right]$$

$$\text{and} \quad \rho/\mu_\mathrm{e} m_\mathrm{H} = \frac{4\pi}{h^3} (2mkT)^{3/2}\, F_{\frac{1}{2}}(\Psi),$$

$$\text{where} \quad F_n(\Psi) \equiv \int_0^\infty \frac{u^n}{e^{u-\Psi} + 1}\, \mathrm{d}u.$$

Write down the equation of state for a partially degenerate electron gas. Show that when $\Psi \ll -1$ the perfect gas law holds. What is the equation of state when $\Psi \gg 1$?

3. Show that the Schwarzschild stability criterion for convection can be written in the form $n + 1 < N(r) + 1$, where $\gamma = (n+1)/n$ is the adiabatic index and $N(r)$ is the local polytropic index defined by $N + 1 = TdP/PdT$.
A box of pure hydrogen is compressed adiabatically. Sketch the behaviour of P with T as T varies between 3,000 K and 30,000 K. Plot the value of the adiabatic index you would deduce from your graph.

What can you say about the convective stability of ionization zones?

4. In the outer layers of a star we can take $L = \text{const}$, $m = \text{const}$ and the opacity can be assumed to be of the form

$$\kappa \propto P^{\alpha-1}/T^{\beta-4},$$

where α and β are constant. Write down, without detailed justification, a differential equation and a boundary condition that serves serve to determine temperature T as a function of pressure P through the outer layers. Investigate the asymptotic behaviour of $T(P)$ at great depth and show that the atmosphere must be convective (except possibly for a thin surface layer) if

$$\alpha > \frac{2}{5}\beta > 0$$

when the adiabatic gradient is 2/5. Discuss the cases

$$\kappa = \text{const}, \ \kappa \propto \rho/T^{3.5} \text{ and } \kappa \propto \rho^{\frac{1}{2}}T^{10}.$$

What is their relevance?
Under what opacity laws would the atmosphere tend to be isothermal at great depth?

5. Derive the Schwarzschild criterion for convective stability when $\gamma = 5/3$.
In a stellar atmosphere radiative transfer theory leads to

$$T^4 = \frac{3}{4}T_{\mathrm{e}}^4\left(\tau + \frac{2}{3}\right),$$

where T_{e} is the effective temperature and T is the actual temperature at optical depth

$$\tau = \int_r^\infty \kappa\rho dr.$$

The atmosphere of the cool star contains negligible mass and is in hydrostatic equilibrium. The opacity within the atmosphere is given by

$$\kappa = \kappa_0 P^{\frac{1}{2}}T^8,$$

where κ_0 is constant. Show that

$$P^{\frac{3}{2}} = \frac{8GM}{R^2\kappa_0 T_{\mathrm{e}}^8}\left[\frac{1}{2} - \frac{1}{3\tau+2}\right].$$

Find the value of $\tau = \tau_{\mathrm{c}}$ at which convection sets in.
If the star is fully convective for $\tau > \tau_{\mathrm{c}}$, so that the equation of state is $P = A\rho^{\frac{5}{3}}$ with A determined at $\tau = \tau_{\mathrm{c}}$, deduce that

$$L \propto R^{98/47}M^{28/47}.$$

[*You may use the fact that for an* $n = \frac{3}{2}$ *polytrope* $R \propto AM^{-\frac{1}{3}}$.]

6. Define the critical luminosity $L_{\mathrm{crit}}(r) = 4\pi cGm/\kappa$.
Show that

$$dP_{\mathrm{g}}/dP_{\mathrm{r}} = \left(L_{\mathrm{crit}} - L_{\mathrm{rad}}\right)/L_{\mathrm{rad}},$$

where L_{rad} is the radiatively transferred luminosity.
What happens if $L_{\mathrm{crit}} < L_{\mathrm{rad}}$ (a) inside the star (b) at the surface?
What is the value of L_{crit} for electron scattering opacity?