1. Apply the method of multiple scales to the Duffing equation
\[
\frac{d^2 u}{dt^2} + u + \epsilon u^3 = 0 \quad \text{with} \quad \epsilon \ll 1
\]
with initial conditions \( \frac{du}{dt} = 0, u = 1 \) at \( t = 0 \) to find the long-time evolution uniformly in \( \epsilon t \leq O(1) \). Repeat your analysis when the cubic term in the Duffing equation is replaced by \( \epsilon (\frac{du}{dt})^3 \).

2. **Stroboscopic method.** A perturbed oscillator satisfies
\[
\frac{d^2 u}{dt^2} + u = \epsilon f \left( \frac{du}{dt}, u, t \right).
\]
By using the method of multiple scales show that the leading-order solution takes the form
\[
u = R(T) \cos(t + \phi(T)),
\]
where \( T = \epsilon t \),
\[
\frac{dR}{dT} = -< f \sin(t + \phi) >, \quad R \frac{d\phi}{dT} = -< f \cos(t + \phi) >,
\]
and \(< .. > \) denotes the average over the fast time period \( 0 < t < 2\pi \). Use these results to re-derive your answers to question 1.

3. Find the leading-order approximation to the general solution for \( x(t; \epsilon) \) and \( y(t; \epsilon) \) satisfying
\[
\frac{d^2 x}{dt^2} + 2\epsilon y \frac{dx}{dt} + x = 0, \quad \frac{dy}{dt} = \frac{1}{2} \epsilon \ln x^2,
\]
which is valid for \( t = \text{ord}(1/\epsilon) \) as \( \epsilon \to 0 \). You may quote the result
\[
\frac{1}{2\pi} \int_0^{2\pi} \ln \cos^2 \theta \, d\theta = -\ln 4.
\]

4. Solve the Mathieu equation
\[
\ddot{y} + (\omega^2 + \epsilon \cos t)y = 0,
\]
for the case when \( \omega = \frac{1}{2} + \epsilon \omega_1 + \ldots \). Identify the stability boundary correct to \( \text{ord}(\epsilon) \).

*Explain why it is necessary to introduce a slow time \( T = \epsilon^2 t \) in order to calculate the stability boundary correct to \( \text{ord}(\epsilon^2) \) and perform the calculation.
5. The function $u(t; \epsilon)$ satisfies the governing equation

$$\frac{d^2 u}{dt^2} - \lambda \epsilon^2 \frac{du}{dt} + \epsilon \gamma u^2 \frac{du}{dt},$$

and the initial conditions

$$u = 2a, \quad \text{and} \quad \frac{du}{dt} = 0 \quad \text{at} \quad t = 0,$$

where $0 < \epsilon \ll 1$, and $\lambda$, $\gamma$, and $a$ are order one constants. By ascertaining at what order of $\epsilon$ a secularity first appears in the regular perturbation expansion for $u(t; \epsilon)$, or otherwise, find a solution for $|u(t)|^2$ that is uniformly valid for large times. If $\lambda > 0$, sketch typical solutions for $|u|^2$ for both $\gamma > 0$ and $\gamma < 0$. Sketch the squared amplitude as a function of time for different values of $a$, with special emphasis on the case $|a| \ll 1$.

6. Find the leading-order approximation which is valid for times $t = \text{ord}(\epsilon^{-1})$ as $\epsilon \to 0$, to the solution $x(t; \epsilon)$ and $y(t; \epsilon)$ satisfying

$$\frac{dx}{dt} + x^2 y \cos t = \epsilon(x - 2x^2),$$

$$\frac{dy}{dt} = \epsilon \left(1 - \frac{\sin t}{x}\right),$$

with $x = 1$ and $y = 0$ at $t = 0$.

7. Use the transformation

$$x(t, \epsilon) = \Re \left[r(\epsilon t, \epsilon) \exp \left(i \int_0^t \sigma(\epsilon q, \epsilon) dq \right) \right],$$

to obtain a higher order approximation correct to $O(\epsilon^2)$ to the equation

$$\ddot{x} + f(\epsilon t)x = 0,$$

where the function $f$ is real.

8. Find the large eigenvalue solutions of the equation

$$y'' + \lambda (1 - x^2)^2 y = 0,$$

subject to $y = 0$ at $x = \pm 1$. At the ends $x = \pm 1$ you will need to use turning point solutions like

$$(1 - x^2)^{1/2} J_{1/4}(\lambda^{1/2}(1 - x^2)^{2/4}),$$

and then use

$$J_{1/4}(z) \sim (2/\pi z)^{1/2} \cos(z - 3\pi/8) \quad \text{as} \quad z \to \infty.$$

9. Sound waves propagating through a slow-varying mean flow satisfy the equations

$$\rho_0 (\tilde{u}_t + (U \tilde{u})_z) = -c_0^2 \tilde{\rho}_z, \quad \tilde{\rho}_t + (U \tilde{\rho})_z = -\rho_0 \tilde{u}_z,$$

where the wavespeed $c_0$ and the undisturbed density $\rho_0$ are constants, $\tilde{u}(z, t)$ and $\tilde{\rho}(z, t)$ are the perturbation velocity and density respectively, and $U(\epsilon z, \epsilon t)$ is the slowly-varying mean flow. By seeking solutions of the form,

$$(\tilde{\rho}, \tilde{u}) = ((A_0, B_0)(\epsilon z, \epsilon t) + \epsilon (A_1, B_1)(\epsilon z, \epsilon t) + \ldots) \exp(i\theta(\epsilon z, \epsilon t)/\epsilon) + \text{c.c.},$$

show that the wave action $E_r/\omega_r$ is conserved, where

$$E_r = \frac{c_0^2 |A_0|^2}{2\rho_0}, \quad \omega_r = \omega - kU.$$