3C7c **Perturbation Methods: Examples Sheet 3** Michaelmas 2022

A star $*$ denotes a question, or part of a question, that should not be done at the expense of unstarred questions (e.g. you might like to miss them out first time through the sheet). You are welcome to use an algebraic manipulator if you think it would help. Corrections, suggestions and comments should be emailed to `S.J.Cowley@maths.cam.ac.uk`.

If you would like questions 2, 4, 7 and 9 marked in advance of the third Examples Class on 23 January 2023, please note the following:

- *the deadline for handing in your work is midnight on Thursday 19 January 2023;*
- *please place your work in the folder in Stephen Cowley's DAMTP pigeonhole in the CMS;*
- *please put your full name and CRSid on your work, and staple (or equivalent) your work together.*

1. Apply the method of multiple scales to the Duffing equation

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0 \quad \text{with} \quad \epsilon \ll 1$$

with initial conditions $du/dt = 0$, $u = 1$ at $t = 0$ to find the long-time evolution uniformly in $\epsilon t \leqslant O(1)$. Repeat your analysis when the cubic term in the Duffing equation is replaced by $\epsilon(du/dt)^3$.

2. *Stroboscopic method.* A perturbed oscillator satisfies

$$\frac{d^2u}{dt^2} + u = \epsilon f\left(\frac{du}{dt}, u, t\right) \quad .$$

By using the method of multiple scales show that the leading-order solution takes the form $u = R(T)\cos(t + \phi(T))$, where $T = \epsilon t$,

$$\frac{dR}{dT} = -< f\sin(t+\phi) > \qquad R\frac{d\phi}{dT} = -< f\cos(t+\phi) > \ ,$$

and $< .. >$ denotes the average over the fast time period $0 < t < 2\pi$. Use these results to re-derive your answers to question 1.

3. Find the leading-order approximation to the general solution for $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\epsilon y\frac{dx}{dt} + x &= 0 \ , \\ \frac{dy}{dt} &= \tfrac{1}{2}\epsilon \ln x^2 \ , \end{aligned}$$

which is valid for $t = \text{ord}(1/\epsilon)$ as $\epsilon \to 0$. You may quote the result

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\cos^2\theta \, d\theta = -\ln 4 \ .$$

4. Solve the Mathieu equation

$$\ddot{y} + (\omega^2 + \epsilon\cos t)y = 0 \ ,$$

for the case when $\omega = \frac{1}{2} + \epsilon\omega_1 + \ldots$. Identify the stability boundary correct to $\text{ord}(\epsilon)$.

*Explain why it is necessary to introduce a slow time $\mathcal{T} = \epsilon^{\frac{3}{2}}t$ in order to calculate the stability boundary correct to $\text{ord}(\epsilon^2)$ and perform the calculation.

5. The function $u(t;\epsilon)$ satisfies the governing equation

$$\frac{d^2u}{dt^2} - \lambda\epsilon^2 t\frac{du}{dt} + u = \epsilon\gamma u^2\frac{du}{dt} ,$$

and the initial conditions

$$u = 2a , \quad \text{and} \quad \frac{du}{dt} = 0 \quad \text{at} \quad t = 0 ,$$

where $0 < \epsilon \ll 1$, and λ, γ and a are order one constants. By ascertaining at what order of ϵ a secularity first appears in the regular perturbation expansion for $u(t;\epsilon)$, or otherwise, find a solution for $|u(t)|^2$ that is uniformly valid for large times. If $\lambda > 0$, sketch typical solutions for $|u|^2$ for both $\gamma > 0$ and $\gamma < 0$. Sketch the squared amplitude as a function of time for different values of a, with special emphasis on the case $|a| \ll 1$.

6. Find the leading-order approximation which is valid for times $t = \text{ord}(\epsilon^{-1})$ as $\epsilon \to 0$, to the solution $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\begin{aligned}\frac{dx}{dt} + x^2 y\cos t &= \epsilon(x - 2x^2) ,\\ \frac{dy}{dt} &= \epsilon\left(1 - \frac{\sin t}{x}\right) ,\end{aligned}$$

with $x = 1$ and $y = 0$ at $t = 0$.

7. Use the transformation

$$x(t,\epsilon) = \Re\left[r(\epsilon t,\epsilon)\exp\left(\mathrm{i}\int^t \sigma(\epsilon q,\epsilon)\,dq\right)\right] ,$$

to obtain a higher order approximation correct to $O(\epsilon^2)$ to the equation

$$\ddot{x} + f(\epsilon t)x = 0 ,$$

where the function f is real.

∗8. Find the large eigenvalue solutions of the equation

$$y'' + \lambda(1 - x^2)^2 y = 0 ,$$

subject to $y = 0$ at $x = \pm 1$. At the ends $x = \pm 1$ you will need to use turning point solutions like

$$(1 - x^2)^{1/2}\mathrm{J}_{1/4}(\lambda^{1/2}(1 - x^2)^2/4),$$

and then use

$$\mathrm{J}_{1/4}(z) \sim (2/\pi z)^{1/2}\cos(z - 3\pi/8) \quad \text{as} \quad z \to \infty.$$

9. Sound waves propagating through a slow-varying mean flow satisfy the equations

$$\rho_0(\tilde{u}_t + (U\tilde{u})_z) = -c_0^2\tilde{\rho}_z , \quad \tilde{\rho}_t + (U\tilde{\rho})_z = -\rho_0\tilde{u}_z ,$$

where the wavespeed c_0 and the undisturbed density ρ_0 are constants, $\tilde{u}(z,t)$ and $\tilde{\rho}(z,t)$ are the perturbation velocity and density respectively, and $U(\epsilon z,\epsilon t)$ is the slowly-varying mean flow. By seeking solutions of the form,

$$(\tilde{\rho},\tilde{u}) = ((A_0,B_0)(\epsilon z,\epsilon t) + \epsilon(A_1,B_1)(\epsilon z,\epsilon t) + \ldots)\exp(\mathrm{i}\theta(\epsilon z,\epsilon t)/\epsilon) + \text{c.c.} ,$$

show that the wave action E_r/ω_r is conserved, where

$$E_r = \frac{c_0^2|A_0|^2}{2\rho_0} , \quad \text{and} \quad \omega_r = \omega - kU .$$