1. Suppose that \( U(\phi) \geq 0 \) and that \( U = 0 \) at one or more vacuum values of \( \phi \).
   
   (a) Show that for a kink satisfying a Bogomolny equation \( \frac{d\phi}{dx} = \pm \frac{dW}{d\phi} \), the static field equation
   \[
   \frac{d^2\phi}{dx^2} = \frac{dU}{d\phi}
   \]
   is automatically satisfied, where \( U(\phi) = \frac{1}{2} (\frac{dW}{d\phi})^2 \).

   (b) Show that the equation \( \frac{d^2\phi}{dx^2} = \frac{dU}{d\phi} \) can be interpreted as the equation for particle motion in the inverted potential \(-U\), where the ‘position’ \( \phi \) is regarded as a function of ‘time’ \( x \).

   (c) Assuming that the vacua of \( U \) are quadratic minima (i.e. with positive second derivative), find the generic form of \( \phi(x) \) as \( \phi \) approaches one of the vacua. Suppose \( U \) has quadratic vacua at \( \phi_1, \phi_2 \) and \( \phi_3 \) in increasing order. Show that there is no static kink connecting \( \phi_1 \) and \( \phi_3 \). (Hint: Think about the interpretation in terms of particle motion.)

2. (Derrick’s theorem for kinks) The energy of a static kink is \( E = E_1 + E_2 \), where
   \[
   E_1 = \int_{-\infty}^{\infty} \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 \, dx, \quad E_2 = \int_{-\infty}^{\infty} U(\phi) \, dx.
   \]
   Show that replacing the kink \( \phi(x) \) by a rescaled field configuration \( \lambda \phi(\lambda x) \), with \( \lambda \) a positive constant, changes the energy to \( E = \lambda E_1 + \frac{1}{\lambda} E_2 \). Deduce that for the kink, \( E_1 = E_2 = \frac{1}{2} E \).

   Deduce that for a kink moving non-relativistically, with field \( \phi(x - vt) \), where \( \phi(x) \) is the static kink, the kinetic energy is \( T = \frac{1}{2} M v^2 \), where \( M \) is the mass of the kink.

3. Starting with the Lagrangian density of the Sine-Gordon theory
   \[
   \mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 - (1 - \cos \beta \phi),
   \]
   derive the Sine-Gordon field equation. Find the function \( W \), as defined in Q.1, and use it to find a static kink solution of the Sine-Gordon theory. Use the Bogomolny bound to find its energy. How many types of kink and antikink are there? How do your results change if \( \phi \) is regarded as an angle in the range \( 0 \leq \phi < 2\pi/\beta \).

4. The Lagrangian density for a complex scalar field \( \phi \) in 1 + 1 dimensions is
   \[
   \mathcal{L} = \frac{1}{2} \left| \frac{\partial \phi}{\partial t} \right|^2 - \frac{1}{2} \left| \frac{\partial \phi}{\partial x} \right|^2 - \frac{1}{2} \lambda^2 (a^2 - |\phi|^2)^2.
   \]
   Verify that the field equation is
   \[
   \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - 2\lambda^2 (a^2 - |\phi|^2) \phi = 0
   \]
and that it has the real kink \( \phi_0(x) = a \tanh \lambda x \) as a solution. Now consider a small pure imaginary perturbation \( \phi(x,t) = \phi_0(x) + i\eta(x,t) \), with \( \eta \) real. Find the linear equation for \( \eta \). By considering \( \eta \) of the form \( \text{sech}(\alpha x)e^{\omega t} \), show that the kink is unstable. Is there a topological argument suggesting that the kink is either stable or unstable?

5. Study the kinks of the \( \phi^8 \) theory, with a triple well potential \( U(\phi) = \frac{1}{2}(1 - \phi^2)^2\phi^4 \). Sketch the potential and determine how many types of kink there are. What are their energies? Find the (implicit) form of the kink interpolating between \( \phi = 0 \) as \( x \to -\infty \) and \( \phi = 1 \) as \( x \to \infty \).