Part III **Prof. N.S. Manton**
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Classical and Quantum Solitons
Examples 1 – Kinks, Abelian Higgs Model

1. Suppose that $U(\phi) \geq 0$ and that $U = 0$ at one or more vacuum values of ϕ.

(a) Show that for a kink satisfying a Bogomolny equation $\frac{d\phi}{dx} = \pm\frac{dW}{d\phi}$, the static field equation

$$\frac{d^2\phi}{dx^2} = \frac{dU}{d\phi}$$

is automatically satisfied, where $U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$.

(b) Show that the equation $\frac{d^2\phi}{dx^2} = \frac{dU}{d\phi}$ can be interpreted as the equation for particle motion in the inverted potential $-U$, where the 'position' ϕ is regarded as a function of 'time' x.

(c) Assuming that the vacua of U are quadratic minima (i.e. with positive second derivative), find the generic form of $\phi(x)$ as ϕ approaches one of the vacua. Suppose U has quadratic vacua at ϕ_1, ϕ_2 and ϕ_3 in increasing order. Show that there is no static kink connecting ϕ_1 and ϕ_3. (Hint: Think about the interpretation in terms of particle motion.)

2. (Derrick's theorem for kinks) The energy of a static kink is $E = E_1 + E_2$, where

$$E_1 = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 dx\,, \qquad E_2 = \int_{-\infty}^{\infty} U(\phi)\,dx\,.$$

Show that replacing the kink $\phi(x)$ by a rescaled field configuration $\phi(\lambda x)$, with λ a positive constant, changes the energy to $E = \lambda E_1 + \frac{1}{\lambda}E_2$. Deduce that for the kink, $E_1 = E_2 = \frac{1}{2}E$.

Deduce that for a kink moving non-relativistically, with field $\phi(x - vt)$, where $\phi(x)$ is the static kink, the kinetic energy is $T = \frac{1}{2}Mv^2$, where M is the mass of the kink.

3. Find the components of the energy-momentum tensor

$$T^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^{\mu}_{\ \nu}\mathcal{L}$$

of a general scalar field theory in $1+1$ dimensions, with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi)$. Show that for a static kink obeying a Bogomolny equation, there is no force acting on any part of the kink.

4. Starting with the Lagrangian density of the Sine-Gordon theory

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - (1 - \cos\beta\phi)\,,$$

derive the Sine-Gordon field equation. Find the function W, as defined in Q.1, and use it to find a static kink solution of the Sine-Gordon theory. Use the Bogomolny bound to find its energy. How many types of kink and antikink are there? How do your results change if ϕ is regarded as an angle in the range $0 \le \phi < 2\pi/\beta$.

5. In Sine–Gordon theory, there is a 2-kink solution

$$\phi(x,t) = 4\tan^{-1}\left[\frac{v\sinh\gamma x}{\cosh\gamma v t}\right], \quad \gamma = (1-v^2)^{-1/2}.$$

Sketch a graph of $v\sinh\gamma x$ for v small and positive, and hence sketch a graph of $\phi(x)$ for (i) $t = 0$ and (ii) $|t|$ large. Estimate the kink separation at closest approach. What are the velocities of the kinks when $|t|$ is large?

6. The Lagrangian density for a complex scalar field ϕ in $1+1$ dimensions is

$$\mathcal{L} = \frac{1}{2}\left|\frac{\partial\phi}{\partial t}\right|^2 - \frac{1}{2}\left|\frac{\partial\phi}{\partial x}\right|^2 - \frac{1}{2}\lambda^2(a^2 - |\phi|^2)^2.$$

Verify that the field equation is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - 2\lambda^2(a^2 - |\phi|^2)\phi = 0$$

and that it has the real kink $\phi_0(x) = a\tanh\lambda a x$ as a solution. Now consider a small pure imaginary perturbation $\phi(x,t) = \phi_0(x) + i\eta(x,t)$, with η real. Find the linear equation for η. By considering η of the form $\mathrm{sech}(\alpha x)e^{\omega t}$, show that the kink is unstable. Is there a topological argument suggesting that the kink is either stable or unstable?

7. Study the kinks of the ϕ^8 theory, with a triple well potential $U(\phi) = \frac{1}{2}(1-\phi^2)^2\phi^4$. Sketch the potential and determine how many types of kink there are. What are their energies? Find the (implicit) form of the kink interpolating between $\phi = 0$ as $x \to -\infty$ and $\phi = 1$ as $x \to \infty$.

8. Consider the abelian Higgs model in the plane, in polar coordinates, and assume the fields are initially smooth.

(a) Show that a smooth gauge transformation has zero net winding on the circle at infinity, and deduce that the winding number of the field, N, is gauge invariant.

(b) Show that by a smooth gauge transformation, the fields can be transformed to a gauge where $a_r = 0$. Show that it is not generally possible to transform to a gauge where $a_\theta = 0$.

9. Verify the covariant Leibniz rule

$$\partial_i(\phi^* D_j\phi) = (D_i\phi)^* D_j\phi + \phi^* D_i D_j\phi,$$

and also the identity $[D_i, D_j]\phi = -if_{ij}\phi$. Use these to complete the derivation of the Bogomolny energy bound and the Bogomolny equations in the abelian Higgs model at critical coupling.