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Easter Term 2019

Classical and Quantum Solitons
Examples 2 – Sine–Gordon Kinks, Vortices and Rational Maps

1. In Sine–Gordon theory, there is a 2-kink solution

$$\phi(x,t) = 4\tan^{-1}\left[\frac{v\sinh\gamma x}{\cosh\gamma vt}\right], \quad \gamma = (1-v^2)^{-1/2}.$$

Sketch a graph of $v\sinh\gamma x$ for v small and positive, and hence sketch a graph of $\phi(x)$ for (i) $t = 0$ and (ii) $|t|$ large. Estimate the kink separation at closest approach. What are the velocities of the kinks when $|t|$ is large?

2. Consider the abelian Higgs model in the plane, in polar coordinates, and assume the fields are initially smooth.

(a) Show that a smooth gauge transformation has zero net winding on the circle at infinity, and deduce that the winding number of the field, N, is gauge invariant.

(b) Show that by a smooth gauge transformation, the fields can be transformed to a gauge where $a_r = 0$. Show that it is not generally possible to transform to a gauge where $a_\theta = 0$.

3. Verify the covariant Leibniz rule

$$\partial_i(\phi^* D_j\phi) = (D_i\phi)^* D_j\phi + \phi^* D_i D_j\phi,$$

and also the identity $[D_i, D_j]\phi = -if_{ij}\phi$. Use these to complete the derivation of the Bogomolny energy bound and the Bogomolny equations in the abelian Higgs model.

4. The energy of static fields in the abelian Higgs model on a general surface Σ with coordinates $y^i : i = 1,2$ and metric $ds^2 = g_{ij}dy^i dy^j$ is

$$E = \int_\Sigma \left\{\frac{1}{4}f_{ij}f_{kl}\, g^{ik}g^{jl} + \frac{1}{2}(D_i\phi)^* D_j\phi\, g^{ij} + \frac{\lambda}{8}(1-\phi^*\phi)^2\right\}\sqrt{\det g}\, dy^1 dy^2,$$

where $\det g$ is the determinant of the metric tensor and g^{ij} is the inverse metric tensor.

Find the expression for E in the Euclidean plane, using polar coordinates r,θ and the appropriate metric. Assuming the fields of a unit winding vortex have the form

$$a_r = 0\,,\ a_\theta = f(r)\,,\ \phi = h(r)e^{i\theta}\,,$$

simplify the expression for E and deduce that the field equations reduce to the coupled ODEs

$$\frac{d^2h}{dr^2} + \frac{1}{r}\frac{dh}{dr} - \frac{1}{r^2}(1-f)^2 h + \frac{\lambda}{2}(1-h^2)h = 0\,,$$
$$\frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr} + (1-f)h^2 = 0\,.$$

5. The Bogomolny equations for a unit winding vortex, with fields as in Q.4, simplify to

$$\frac{dh}{dr} - \frac{1}{r}(1-f)h = 0\,,$$
$$\frac{df}{dr} - \frac{r}{2}(1-h^2) = 0\,.$$

By differentiating, verify that if these are satisfied, then so are the second order ODEs in Q.4, with $\lambda = 1$.

By eliminating f, find the radial form of the Taubes equation for $u = 2\log h$.

6. Consider a surface Σ with real coordinates y^1, y^2 and metric $ds^2 = \Omega(y^1, y^2)((dy^1)^2 + (dy^2)^2)$. (Actually, by a suitable choice of coordinates, any metric on a surface can be put in this form locally. Ω is called the conformal factor.) Find the simplified form of the energy expression E in Q.4 for this metric. When $\lambda = 1$, a variant of the Bogomolny rearrangement leads to Bogomolny equations where the usual $\frac{1}{2}$ is replaced by $\frac{\Omega}{2}$ in the second equation, and the resulting Taubes equation (ignoring the delta functions) is

$$\nabla^2 u - \Omega e^u + \Omega = 0\,.$$

The general formula for the Gaussian curvature of Σ is $K = -\frac{1}{2\Omega}\nabla^2 \log\Omega$. Find the constant value of K for which the change of variable $u = \sigma - \log\Omega$ reduces the Taubes equation to the Liouville equation

$$\nabla^2\sigma - e^\sigma = 0\,.$$

Verify that the appropriate value of K occurs for $\Omega = \frac{8}{(1-r^2)^2}$, where $r^2 = (y^1)^2 + (y^2)^2$ and $r < 1$. (This is the conformal factor for the Poincaré disc model of the hyperbolic plane.)

Show that a solution of Liouville's equation is

$$\sigma = \log\left(\frac{32r^2}{(1-r^4)^2}\right)\,,$$

and by combining σ and Ω appropriately, show that on the Poincaré disc there is a 1-vortex solution for which the magnitude of the Higgs field is

$$|\phi| = \frac{2r}{1+r^2}\,.$$

Check that this satisfies the required boundary conditions.

7. On the surface Σ with real coordinates and metric as in Q.6, introduce a complex coordinate $z = y^1 + iy^2$ and its complex conjugate $\bar{z}$. Show that

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)\,, \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)\,.$$

Let $a_z = \frac{1}{2}(a_1 - ia_2)$ and $a_{\bar{z}} = \frac{1}{2}(a_1 + ia_2)$ and find the relation between $f_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z$ and $B = f_{12}$. Write the metric and area element on the surface in terms of dz and $d\bar{z}$.

Rewrite the Bogomolny equations using $D_{\bar{z}}\phi$ and $f_{z\bar{z}}$. Solve the first Bogomolny equation, and rederive the Taubes equation. [You will need to relate $\partial_z\partial_{\bar{z}}$ to the Laplacian.]

8. The Riemann sphere coordinate on S^2 is $z = \tan\frac{\vartheta}{2}\, e^{i\varphi}$, where ϑ, φ are standard polar coordinates. Show that the standard metric and area 2-form are

$$ds^2 = \frac{4dzd\bar{z}}{(1+|z|^2)^2}\,, \quad dA = 2i\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}\,,$$

and hence that $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ are isothermal coordinates.

Let $w = R(z)$ be a rational map from S^2 to S^2. Find the pull-back of the area form on the w-sphere by R and deduce that R is orientation-preserving.

9. Let (x_1, x_2, x_3) be Cartesian coordinates in $\mathbb{R}^3$, and S^2 the unit sphere. Find the formula for z on the unit sphere in terms of (x_1, x_2, x_3). Find the values of z corresponding to $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

Consider a cube in its standard orientation centred at the origin, with vertices on the unit sphere. Split the vertices into two sets of four, with each set being the vertices of a tetrahedron. Find the Cartesian coordinates and hence the value of z at these vertices.

For each tetrahedron, find the monic quartic polynomial (with leading term z^4) whose zeros are the four vertices of the tetrahedron. Find the monic order-8 polynomial whose zeros are all eight vertices of the cube.

10. Consider a rational map of degree d,

$$R(z) = \frac{p(z)}{q(z)}\,,$$

where p and q are (generic) polynomials of degree d. The Wronskian of R is the polynomial $W(z) = p'(z)q(z) - q'(z)p(z)$. Verify that the derivative of R vanishes where W vanishes. Show that W is a polynomial of degree $2d-2$. Show that under a Möbius transformation,

$$R \to \frac{\alpha R + \beta}{\gamma R + \delta}\,,$$

the zeros of the Wronskian are unaffected.

How can you interpret a Wronskian that has degree less than $2d - 2$? Investigate your interpretation for the map $R(z) = z^d$.

11. Consider the 1-parameter family of degree-3 rational maps

$$R(z) = \frac{\sqrt{3}az^2 - 1}{z(z^2 - \sqrt{3}a)}$$

where a is complex. Find the Wronskian of these maps, and find the values of a for which the Wronskian is tetrahedrally symmetric.