Part III **Prof. N.S. Manton**
Easter Term 2019

Classical and Quantum Solitons
Examples 3 – Skyrmions

1. The sigma model in 2+1 dimensions has Lagrangian

$$L = \int \partial_\mu \phi \cdot \partial^\mu \phi \, d^2x \,,$$

where the field $\phi = (\phi_1, \phi_2, \phi_3)$ is subject to the constraint $\phi \cdot \phi = 1$. Using a Lagrange multiplier, show that the field equation can be expressed as

$$\partial_\mu \partial^\mu \phi + (\partial_\mu \phi \cdot \partial^\mu \phi)\phi = 0 \,.$$

2. Starting from the Skyrme Lagrangian with massless pions,

$$L = \int \Big\{ -\frac{1}{2}\mathrm{Tr}(R_\mu R^\mu) + \frac{1}{16}\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \Big\} \, d^3x \,,$$

derive the field equation

$$\partial_\mu \Big(R^\mu + \frac{1}{4}[R_\nu, [R^\nu, R^\mu]] \Big) = 0 \,.$$

[Hint: A variation of U has the form $U \to (1 + \varepsilon X)U$ with X in the Lie algebra of SU(2).] Find the linearized field equation for the pion fields, when U is close to 1_2.

3. Write down the energy integral for static Skyrme fields, including the pion mass term. Find the form of Derrick's theorem satisfied by static Skyrmions.

4. Show that the 3-volume of a 3-sphere of unit radius is $2\pi^2$. Calculate the 4-volume of the 4-ball of unit radius.

5. For the $B = 1$ hedgehog Skyrmion, with profile function $f(r)$ (and energy as given in the lectures), find the second order ODE satisfied by f. Find the linearized form of the ODE for large r, and hence the asymptotic form of f. What is the asymptotic form of $U(\mathbf{x})$?

6. Show that if R is a complex (stereographic) Riemann sphere coordinate then the corresponding Cartesian unit vector on S^2 is

$$\hat{\mathbf{n}}_R = \frac{1}{1 + |R|^2}(R + \bar{R}, -i(R - \bar{R}), 1 - |R|^2) \,.$$

7. Consider the rational map approximation, as defined in lectures. Show that the Skyrme energy, for massless pions, is

$$E = \int_0^\infty \Big\{ f'^2 + 2B\frac{\sin^2 f}{r^2}(1 + f'^2) + \mathcal{I}\frac{\sin^4 f}{r^4} \Big\} 4\pi^2 r^2 \, dr \,,$$

where B is the baryon number and $\mathcal{I}$ an angular integral only involving the rational map.

[Hints: Use the area elements on the domain and target 2-spheres of the rational map to find the strain eigenvalues in the angular directions. Show that the part of the energy integral quadratic in angular strains can be reexpressed in terms of the degree of the rational map and hence the baryon number.]

8. Let G denote the symmetry group SO(3) $\times$ SO(3) of rotations and isorotations of the Skyrme model. The $B = 2$ Skyrmion has a discrete end-over-end symmetry. After fixing the Skyrmion's orientation, find the combination $g \in G$ of rotation and isorotation that corresponds to this symmetry. Find a simple curve connecting the identity of G to g, and consider how the action of its group elements produces a closed loop of Skyrme field configurations (actually Skyrmions). Is this loop contractible or not? Hence write down the constraint on the quantum states Ψ of the Skyrmion associated with the symmetry g.

[Hint: Consider deforming the $B = 2$ Skyrmion into a chain of two separated $B = 1$ Skyrmions.]

9. Find the Möbius transformation for a 120° rotation about the $(1, 1, 1)$ axis in space. [Hint: Acting on z it sends $0 \to 1 \to i \to 0$.] Find the effect of this transformation on the rational map $R_4(z)$ with octahedral symmetry given in lectures. Hence find the corresponding constraint on the rotational/isorotational wavefunction of the $B = 4$ Skyrmion. Find the constraint corresponding to the 90° rotational symmetry about the x_3-axis. [Hint: The FR factors are $+1$ for both these – try to verify this.]

10. Determine the relations between baryon number B and the 3rd component of isospin I_3, and the proton and neutron numbers Z and N of a nucleus.

Estimate how the spin and isospin moments of inertia of a Skyrmion increase with baryon number B, for large B. Hence estimate the difference in energy betweeen the lowest spin 0 and spin 2 states of an even-even nucleus, and the asymmetry energy of a nucleus (the term proportional to $(Z - N)^2$).