Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk. Starred questions may if you wish be handed in to your supervisor for feedback prior to the class.

1. Show directly that if \( \phi(x) \) satisfies the Klein-Gordon equation, then \( \phi(\Lambda^{-1}x) \) also satisfies this equation for any Lorentz transformation \( \Lambda \).

2. The motion of a complex field \( \psi(x) \) is governed by the Lagrangian density

\[
\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2.
\]

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

\[
\delta \psi = i\alpha \psi, \quad \delta \psi^* = -i\alpha \psi^*.
\]

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by \( \psi \).

3. Verify that the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a
\]

for a triplet of real fields \( \phi_a \), where \( a \in \{1, 2, 3\} \) is invariant under the infinitesimal \( SO(3) \) rotation by \( \theta \)

\[
\phi_a \rightarrow \phi_a + \theta \epsilon^{abc} \eta_b \phi_c
\]

where \( \eta_a \) is a unit vector. Compute the Noether current \( j^\mu \). Deduce that the three quantities

\[
Q_a = \int d^3x \epsilon^{abc} \dot{\phi}_b \phi_c
\]

are all conserved and verify this directly using the field equations satisfied by \( \phi_a \).

4. A Lorentz transformation \( x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \) is such that it preserves the Minkowski metric \( \eta_{\mu\nu} \), meaning that \( \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu \) for all \( x \). Show that this implies that

\[
\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma_\mu \Lambda^\tau_\nu.
\]

Use this result to show that an infinitesimal transformation of the form

\[
\Lambda^\mu_\nu = \delta^\mu_\nu + \alpha \omega^\mu_\nu
\]

is a Lorentz transformation when \( \omega^{\mu\nu} \) is antisymmetric: i.e. \( \omega^{\mu\nu} = -\omega^{\nu\mu} \) (\( \alpha \) is considered to be infinitesimal).
Write down the matrix form for $\omega^{\mu}_{\nu}$ that corresponds to a rotation through an infinitesimal angle $\theta$ about the $x^3$-axis. Do the same for a boost along the $x^1$-axis by an infinitesimal velocity $v$.

5. Consider the infinitesimal form of the Lorentz transformation derived in the previous question: $x^{\mu} \rightarrow x^{\mu} + \alpha \omega^{\mu}_{\nu} x^\nu$. Show that a scalar field transforms as

$$\phi(x) \rightarrow \phi'(x) = \phi(x) - \alpha \omega^{\mu}_{\nu} x^\nu \partial_\mu \phi(x)$$

and hence show that the variation of the Lagrangian density is a total derivative

$$\delta \mathcal{L} = -\alpha \partial_\mu (\omega^{\mu}_{\nu} x^\nu \mathcal{L}).$$

Using Noether’s theorem, deduce the existence of the conserved current

$$j^\mu = -\omega^\rho_{\nu} [T^\mu_\rho x^\nu].$$

The three conserved charges arising from spatial rotational invariance define the total angular momentum of the field. Show that these charges are given by

$$Q_i = \frac{1}{2} \epsilon_{ijk} \int d^3 x \left( x^j T^{0k} - x^k T^{0j} \right).$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \int d^3 x \left( x^i T^{00} \right) = \text{constant}$$

and interpret this equation.

6. Maxwell’s Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu$ is the 4-vector potential. Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \xi,$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on $x$.

Using Noether’s theorem, and the spacetime translational invariance of the action, to construct the energy momentum tensor $T^{\mu\nu}$ for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu} \partial_\rho A^\nu.$$

Show that this object also defines four currents. Moreover, show that it is symmetric, gauge invariant and traceless.
7. The Lagrangian density for a massive vector field $C_\mu$ is given by

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 C_\mu C^\mu, \]

where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. Derive the equations of motion and show that when $m \neq 0$ they imply

\[ \partial_\mu C^\mu = 0. \]

Further show that $C_0$ can be eliminated completely in terms of other fields by

\[ \partial_i \partial^i C_0 + m^2 C_0 = \partial^i \dot{C}_i. \]  \hspace{1cm} (1)

Construct the canonical momenta $\Pi_i$ conjugate to $C_i$ where $i \in \{1, 2, 3\}$ and show that the canonical momentum conjugate to $C_0$ is vanishing. Construct the Hamiltonian density $H$ in terms of $C_0, C_i$ and $\Pi_i$ (NB: don’t be concerned that the canonical momentum for $C_0$ is vanishing. $C_0$ is non-dynamical; it is determined entirely in terms of the other fields using Eq. (1)).

8. A class of interesting theories is invariant under the simultaneous scaling of all lengths by

\[ x^\mu \rightarrow (x')^\mu = \lambda x^\mu \text{ and } \phi(x) \rightarrow \phi'(x) = \lambda^{-D} \phi(\lambda^{-1} x). \]  \hspace{1cm} (2)

Here, $D$ is called the scaling dimension of the field. Consider the action for a real scalar field given by

\[ S = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g \phi^p. \right) \]

Find the scaling dimension $D$ such that the derivative terms remain invariant. For what values of $m$ and $p$ is the scaling in Eq. (2) a symmetry of the theory? How do these conclusions change for a scalar field living in an $(n+1)$-dimensional spacetime instead of a $3 + 1$ dimensional spacetime?

In $3 + 1$ dimensions, use Noether’s theorem to construct the conserved current $D^\mu$ associated with scaling invariance.