1. Consider a real scalar field with Lagrangian
\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2. \] (1)

Show that, after normal ordering, the conserved four-momentum \( P^\mu = \int d^3 x \ T^{0\mu} \) takes the operator form
\[ P^\mu = \int \frac{d^3 p}{(2\pi)^3} p^\mu a^\dagger_\mathbf{p} \mathbf{a}_\mathbf{p} \] (2)

where \( p^0 = E_\mathbf{p} \) in this expression. From Eq. (2), verify that if \( \phi(x) \) is now in the Heisenberg picture, then
\[ [P^\mu, \phi(x)] = -i \partial^\mu \phi(x). \]

2. Show that in the Heisenberg picture,
\[ \dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2 \phi(x) - m^2 \phi(x). \]

Hence show that the operator \( \phi(x) \) satisfies the Klein-Gordon equation.

3. Let \( \phi(x) \) be a real scalar field in the Heisenberg picture. Show that the relativistically normalised states \( |p\rangle = \sqrt{2E_p a^\dagger_\mathbf{p} |0\rangle} \) satisfy
\[ \langle 0 | \phi(x) | p \rangle = e^{-i p \cdot x}. \]

4. In Example Sheet 1, you showed that the classical angular momentum of a field is given by
\[ Q_i = \frac{1}{2} \epsilon_{ijk} \int d^3 x \left( x^j T^{0k} - x^k T^{0j} \right). \]

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq. (1). Show that, after normal ordering, the quantum operator \( Q_i \) can be written as
\[ Q_i = \frac{i}{2} \epsilon_{ijk} \int \frac{d^3 p}{(2\pi)^3} a^\dagger_\mathbf{p} \left( p^j \frac{\partial}{\partial p^k} - p^k \frac{\partial}{\partial p^j} \right) a_\mathbf{p}. \]

Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state \( |\mathbf{p}\rangle \) has zero angular momentum in its rest frame).

5. Show that the time ordered product \( T(\phi(x_1)\phi(x_2)) \) and the normal ordered product \( :\phi(x_1)\phi(x_2): \) are both symmetric under the interchange of \( x_1 \) and \( x_2 \). Deduce that the Feynman propagator \( \Delta_F(x_1 - x_2) \) has the same symmetry property.
6. Verify Wick’s theorem for the case of three scalar fields:

\[ T(\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3) : + \phi(x_1)\Delta_F(x_2-x_3) + \phi(x_2)\Delta_F(x_3-x_1) + \phi(x_3)\Delta_F(x_1-x_2). \]

7. Examine \( \langle 0 | S | 0 \rangle \) to order \( \lambda^2 \) in \( \phi^4 \) theory. Identify the different contributions arising from an application of Wick’s theorem. Confirm that to order \( \lambda^2 \), the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

\[ \langle 0 | S | 0 \rangle = \exp\left( \sum \right) \]

8. Consider the theory of a complex scalar \( \psi \) and a real scalar \( \phi \) with Lagrangian

\[ L = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \mu^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi - h |\psi|^4 - k \phi^3 - l \phi(\partial_\mu \psi)(\partial^\mu \psi^*). \]

Draw and write down momentum-space Feynman rules for the propagators and interactions of this theory. What are the mass dimensions of \( g, h, k, l \)? From these, identify a property of the theory.

9. Consider the scalar Yukawa theory given by the Lagrangian

\[ L = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \mu^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi. \]

In meson decay \( \phi \to \psi \bar{\psi} \), assuming that \( m > 2\mu \), show that to lowest order in \( g^2 \) that the decay width is

\[ \Gamma = \frac{|g|^2}{16\pi m} \sqrt{1 - 4\mu^2/m^2}. \]

Does this make sense, dimensionally?