

3P1c **Quantum Field Theory: Example Sheet 3** Michaelmas 2024

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1. The chiral representation of the Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

- 2*: Show that if $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then

$$[\gamma^\mu \gamma^\nu, \gamma^\rho \gamma^\sigma] = 2\eta^{\nu\rho} \gamma^\mu \gamma^\sigma - 2\eta^{\mu\rho} \gamma^\nu \gamma^\sigma + 2\eta^{\nu\sigma} \gamma^\rho \gamma^\mu - 2\eta^{\mu\sigma} \gamma^\rho \gamma^\nu.$$

Show further that the matrices $S^{\mu\nu} := \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}(\gamma^\mu \gamma^\nu - \eta^{\mu\nu})$ form a representation of the Lie algebra of the Lorentz group.

3. Using just the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (without reference to a particular representation) and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_\mu \gamma^\mu$ and $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, prove the following results:

- $\text{Tr}\gamma^\mu = 0$
- $\text{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$
- $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0$
- $(\gamma^5)^2 = 1$
- $\text{Tr}\gamma^5 = 0$
- $\not{p}\not{q} = 2p \cdot q - \not{q}\not{p} = p \cdot q + 2S^{\mu\nu} p_\mu q_\nu$
- $\text{Tr}(\not{p}\not{q}) = 4p \cdot q$
- $\text{Tr}(\not{p}_1 \dots \not{p}_n) = 0$ if n is odd
- $\text{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)]$
- $\text{Tr}(\gamma^5 \not{p}_1 \not{p}_2) = 0$
- $\gamma_\mu \not{p} \gamma^\mu = -2\not{p}$
- $\gamma_\mu \not{p}_1 \not{p}_2 \gamma^\mu = 4p_1 \cdot p_2$
- $\gamma_\mu \not{p}_1 \not{p}_2 \not{p}_3 \gamma^\mu = -2\not{p}_3 \not{p}_2 \not{p}_1$
- $\text{Tr}(\gamma^5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4i \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$

4* In the chiral representation, the plane-wave solutions to the Dirac equation are

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \vec{\sigma}} \xi^s \\ \sqrt{p \cdot \vec{\sigma}} \xi^s \end{pmatrix} \quad \text{and} \quad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \vec{\sigma}} \xi^s \\ -\sqrt{p \cdot \vec{\sigma}} \xi^s \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s , with $s \in \{1, 2\}$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that

$$\begin{aligned} u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) &= 2p_0 \delta^{rs} \\ \bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) &= 2m \delta^{rs} \end{aligned}$$

and similarly,

$$\begin{aligned} v^r(\vec{p})^\dagger \cdot v^s(\vec{p}) &= 2p_0 \delta^{rs} \\ \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) &= -2m \delta^{rs}. \end{aligned}$$

Show also that the orthogonality condition between u and v is

$$\bar{u}^s(\vec{p}) \cdot v^r(\vec{p}) = 0,$$

while taking the inner product using \dagger requires an extra minus sign

$$u^s(\vec{p})^\dagger \cdot v^r(-\vec{p}) = 0.$$

5. Using the same notation as Question 4, show that

$$\begin{aligned} \sum_{s=1}^2 u^s(\vec{p}) \bar{u}^s(\vec{p}) &= \not{p} + m, \\ \sum_{s=1}^2 v^s(\vec{p}) \bar{v}^s(\vec{p}) &= \not{p} - m, \end{aligned}$$

where, rather than being contracted, the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

6. The Fourier decomposition of the Dirac field operator $\psi(x)$ and the hermitian conjugate field $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^2 [b_p^s u^s(\vec{p}) e^{i\vec{p} \cdot \vec{x}} + c_p^{s\dagger} v^s(\vec{p}) e^{-i\vec{p} \cdot \vec{x}}], \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^2 [b_p^{s\dagger} u^s(\vec{p})^\dagger e^{-i\vec{p} \cdot \vec{x}} + c_p^s v^s(\vec{p})^\dagger e^{i\vec{p} \cdot \vec{x}}]. \end{aligned}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \{b_p^r, b_q^{s\dagger}\} &= (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}), \\ \{c_p^r, c_q^{s\dagger}\} &= (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}), \end{aligned}$$

with all other anticommutators vanishing. Show that these imply that the field and its conjugate field satisfy the anti-commutation relations

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\} = 0, \\ \{\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\} &= \delta_{\alpha\beta} \delta^{(3)}(\vec{x} - \vec{y}). \end{aligned}$$

7. Using the results of Question 6, show that the quantum Hamiltonian

$$H = \int d^3x \bar{\psi}(-i\gamma^i \partial_i + m)\psi$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \sum_{r=1}^2 [b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r + c_{\vec{p}}^{r\dagger} c_{\vec{p}}^r].$$

8. The Lagrangian density for a fermionic Yukawa theory is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - \lambda\phi\bar{\psi}\psi.$$

Consider $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$ scattering, with initial and final states given by,

$$\begin{aligned} |i\rangle &= \sqrt{4\omega_p\omega_q} b_{\vec{p}}^{s\dagger} c_{\vec{q}}^{r\dagger} |0\rangle \\ |f\rangle &= \sqrt{4\omega_{p'}\omega_{q'}} b_{\vec{p}'}^{s'\dagger} c_{\vec{q}'}^{r'\dagger} |0\rangle. \end{aligned}$$

State the Feynman rules for this theory. Show, using Feynman rules, that the amplitude is given by

$$\mathcal{A} = -(-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})][\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q}')] }{t - \mu^2} - \frac{[\bar{v}^r(\vec{q}) \cdot u^s(\vec{p})][\bar{u}^{s'}(\vec{p}') \cdot v^{r'}(\vec{q}')] }{s - \mu^2} \right).$$

where $t = (p-p')^2$ and $s = (p+q)^2$. Draw the two Feynman diagrams that correspond to these two terms.