3P1d **Quantum Field Theory: Example Sheet 4** Michaelmas 2024

Corrections and suggestions should be emailed to `ac2553@cam.ac.uk`.

1. The Lagrangian density for a pseudoscalar Yukawa interaction is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\gamma^5\psi\,. \tag{1}$$

where $\gamma^5 := -i\gamma^0\gamma^1\gamma^2\gamma^3$.

Derive the amplitude for $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering by using the LSZ formula; in particular define asymptotic states and identify the appropriate correlation function that controls the amplitude.

State the appropriate Schwinger-Dyson equations for this theory. Evaluate explicitly the four-point function appearing in $\psi\bar{\psi} \to \psi\bar{\psi}$ to order λ^2. Identify the components that are connect and disconnected. With this, evaluate explicitly the connected amplitude using LSZ at order λ^2.

From these findings, write down the appropriate Feynman rules, and re-derive $\psi\bar{\psi} \to \psi\bar{\psi}$ to order λ^2 using them.

Based on your findings, apply Feynman rules to evaluate the connected contributions to the amplitude $\psi\psi \to \psi\psi$ at order λ^2.

2*. Consider a theory with three real scalar field ϕ_i, $i = 1, 2, 3$, each of them with mass m_i^2. The three scalars interact via

$$\mathcal{L}_{\text{int}} = \lambda\phi_1(\partial_\mu\phi_2)(\partial^\mu\phi_3)\ . \tag{2}$$

Derive the Schwinger-Dyson equations for this theory. State clearly what are your assumptions .

What are the Feynman rules for this theory? You could derive them by considering a simple process, such as $\phi_1\phi_2 \to \phi_3$, and evaluate it via LSZ and Schwinger-Dyson.

3. By defining an appropriate covariant derivative write down a gauge invariant Lagrangian for a complex scalar field coupled to the electromagnetic field (scalar QED).

Deduce Feynman rules for the interaction vertices of this theory by using LSZ formula (i.e. define appropriate asymptotic states) and the Schwinger-Dyson equation. You may use your findings from problem 2.

4. In the Coulomb gauge we have

$$\vec{A}(x) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}} \left(\vec{\varepsilon}_\lambda(k) a_\lambda(k) e^{-ikx} + \vec{\varepsilon}_\lambda(k) a_\lambda^\dagger(k) e^{ikx} \right) \ . \tag{3}$$

Using this, together with commutations relations of $a_\lambda^\dagger(k)$ and $a_\lambda(k)$ and completion relations for $\vec{\varepsilon}_\lambda(k)$, show that

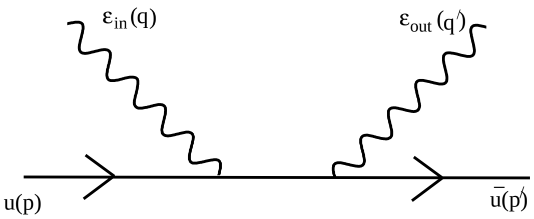
$$\langle \mathrm{T} A^i(x) A^j(y) \rangle = \frac{1}{i} \Delta^{ij}(x-y) \ , \tag{4}$$

where

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2+i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \ . \tag{5}$$

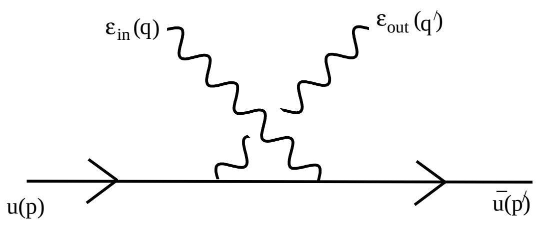
In particular you need to explain the $i\epsilon$ in the propagator.

5*. Consider Compton scattering in which a photon and an electron scatter off each other. Let the incoming photon have polarisation vector $\epsilon^\mu_{\rm in}$ and the outgoing photon have polarisation $\epsilon^\mu_{\rm out}$. Use the Feynman rules to derive the following amplitude associated to the lowest order diagram,



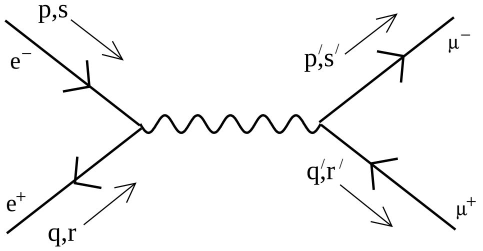
$$= i(-ie)^2 \bar{u}^{r'}(\vec{p}') \not{\epsilon}_{\rm out} \frac{\not{p} + \not{q} + m}{s - m^2} \not{\epsilon}_{\rm in} u^s(\vec{p}). \tag{6}$$

where $s = (p+q)^2$. Also, compute the contribution from the diagram



The complete amplitude at order e^2 is the sum of these two contributions. Show that the total amplitude vanishes if $\epsilon_{\rm in}$ is replaced by the incoming photon momentum q. Check that the same holds true if $\epsilon_{\rm out}$ is replaced by q'.

6. Use the Feynman rules to show that the reduced QED amplitude $\mathcal{A}$ for $e^+e^- \to \mu^+\mu^-$ is given at lowest order in e by



$$= (-ie)^2 \frac{[\bar{v}_e^r(\vec{q})\gamma_\mu u_e^s(\vec{p})][\bar{u}_m^{s'}(\vec{p}')\gamma^\mu v_m^{r'}(\vec{q}')]}{(p+q)^2}, \quad (7)$$

where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons, respectively. Compute the spin-summed/averaged squared matrix element,

$$\mathcal{P} := \frac{1}{4} \sum_{s,r,s',r'=1}^{2} |\mathcal{A}|^2$$

Compare your answer with spin-summed/averaged squared matrix element you would obtain for $e^-\mu^- \to e^-\mu^-$. In particular, what are the appropriate exchanges of momenta such that the two answers agree? (*Note:* This is known as crossing symmetry.)

Returning to $e^+e^- \to \mu^+\mu^-$, working in the center of momentum frame and in the approximation $m_e = 0$, show that

$$\mathcal{P} = e^4 \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right]$$

where E is the energy of each incident particle and θ is the scattering angle.