Symmetries, Fields and Particles. Examples 1.

1. $O(n)$ consists of $n \times n$ real matrices $M$ satisfying $M^T M = I$. Check that $O(n)$ is a group. $U(n)$ consists of $n \times n$ complex matrices $U$ satisfying $U^\dagger U = I$. Check similarly that $U(n)$ is a group.

Verify that $O(n)$ and $SO(n)$ are the subgroups of real matrices in, respectively, $U(n)$ and $SU(n)$. By considering how $U(n)$ matrices act on vectors in $\mathbb{C}^n$, and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, show that $U(n)$ is a subgroup of $SO(2n)$.

2. Show that for matrices $M \in O(n)$, the first column of $M$ is an arbitrary unit vector, the second is a unit vector orthogonal to the first, \ldots, the $k$th column is a unit vector orthogonal to the span of the previous ones, etc. Deduce the dimension of $O(n)$. By similar reasoning, determine the dimension of $U(n)$.

Show that any column of a unitary matrix $U$ is not in the (complex) linear span of the remaining columns.

3. Consider the real $3 \times 3$ matrix,

$$R(n, \theta)_{ij} = \cos \theta \delta_{ij} + (1 - \cos \theta)n_in_j - \sin \theta \epsilon_{ijk}n_k$$

where $n = (n_1, n_2, n_3)$ is a unit vector in $\mathbb{R}^3$. Verify that $n$ is an eigenvector of $R(n, \theta)$ with eigenvalue one. Now choose an orthonormal basis for $\mathbb{R}^3$ with basis vectors $\{n, m, \tilde{m}\}$ satisfying,

$$m \cdot m = 1, \quad m \cdot n = 0, \quad \tilde{m} = n \times m.$$

By considering the action of $R(n, \theta)$ on these basis vectors show that this matrix corresponds to a rotation through an angle $\theta$ about an axis parallel to $n$ and check that it is an element of $SO(3)$.

4. Show that the set of matrices

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 - |\beta|^2 = 1$ forms a group. How would you check that it is a Lie group? Assuming that it is a Lie group, determine its dimension. By splitting $\alpha$ and $\beta$ into real and imaginary parts, consider the group manifold as a subset of $\mathbb{R}^4$ and show that it is non-compact. You may use the fact that a compact subset $S$ of $\mathbb{R}^n$ is necessarily bounded; in other words there exists $B > 0$ such that $|x| < B$ for all $x \in S$.

5. Show that any SU(2) matrix $U$ can be expressed in the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$
with $|\alpha|^2 + |\beta|^2 = 1$. Deduce that an alternative form for an SU(2) matrix is

$$U = a_0 I + i a \cdot \sigma$$

with $(a_0, a)$ real, $\sigma$ the Pauli matrices, and $a_0^2 + a \cdot a = 1$. Using the second form, calculate the product of two SU(2) matrices.

6. Consider a real vector space $V$ with product $*: V \times V \to V$. The product is bilinear and associative. In other words, for all elements $X, Y, Z \in V$ and scalars $\alpha, \beta \in \mathbb{R}$, we have

$$(\alpha X + \beta Y) * Z = \alpha X * Z + \beta Y * Z, \quad Z * (\alpha X + \beta Y) = \alpha Z * X + \beta Z * Y$$

and also $(X * Y) * Z = X * (Y * Z)$. Define the bracket of two vectors $X$ and $Y \in V$ as the commutator,

$$[X, Y] = X * Y - Y * X$$

Show that, equipped with this bracket, $V$ becomes a Lie algebra.

7. Verify that the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{R}$$

forms a matrix Lie group, $G$. What is the underlying manifold of $G$? Is the group abelian? Find the Lie algebra, $L(G)$, and calculate the bracket of two general elements of it. Is the Lie algebra simple?

8. A useful basis for the Lie algebra of $\text{GL}(n)$ consists of the $n^2$ matrices $T^i_j$ $(1 \leq i, j \leq n)$, where $(T^i_j)_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Find the structure constants in this basis.

9. Let $\exp iH = U$. Show that if $H$ is hermitian then $U$ is unitary. Show also, that if $H$ is traceless then $\det U = 1$. How do these results relate to the theorem that the exponential map $X \to \exp X$ sends $L(G)$, the Lie algebra of $G$, to $G$?