Examples Sheet 1

1. $O(n)$ consists of $n \times n$ real matrices M satisfying $M^T M = I$ whereas $U(n)$ consists of $n \times n$ complex matrices U satisfying $U^\dagger U = I$. Check that $O(n)$ is a group. Check similarly that $U(n)$ is a group. Verify that the subset of all real matrices in $U(n)$ forms the group $O(n)$ and, similarly, that the subset of all real matrices in $SU(n)$ forms the group $SO(n)$. By considering the action of $U(n)$ on $\mathbb{C}^n$, and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, show that $U(n)$ is a subgroup of $O(2n)$.

2. Show that for matrices $M \in O(n)$, the first column of M is an arbitrary unit vector, the second is a unit vector orthogonal to the first, ..., the k^{th} column is a unit vector orthogonal to the span of the previous ones, etc. Deduce the dimension of $O(n)$. By similar reasoning, determine the dimension of $U(n)$.

3. (a) Show that a general element of $SU(2)$ can be written as

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

where α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

(b) Deduce that an alternative form for an $SU(2)$ matrix is

$$U = a_0 I + i\mathbf{a} \cdot \boldsymbol{\sigma}$$

with $(a_0, \mathbf{a})$ real, $\boldsymbol{\sigma}$ the Pauli matrices, and $a_0^2 + \mathbf{a} \cdot \mathbf{a} = 1$.

(c) Using the second form, calculate the product of two $SU(2)$ matrices.

4. The symplectic group is formed of matrices M which satisfy $M^\mathsf{T} J M = J$:

$$Sp(2n, \mathbb{R}) := \left\{ M \in GL(2n, \mathbb{R}) \;\middle|\; M^\mathsf{T} \Omega M = \Omega \right\}.$$

where

$$J = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

or

$$J_{ij} = \begin{cases} \delta_{i+1,j}\,, & i \text{ odd} \\ -\delta_{i-1,j}\,, & i \text{ even}\,. \end{cases}$$

[This simplifies some steps compared to using the Ω defined in lecture.]

(a) Show that Pf $J = 1$.

(b) Show that $\det M = 1$ for $M \in Sp(2n, \mathbb{R})$.

5. A Lie group has group elements $g(x)$ depending on group parameters x^r, with $g(0) = e$, the identity, and under group multiplication $g(x)g(y) = g(\varphi(x, y))$ for some $\varphi^r(x, y)$. Let $g(x)^{-1} = g(\bar{x})$ where $\varphi^r(\bar{x}, x) = 0$.

   (a) Why must $\varphi^r(x, 0) = x^r$, $\varphi^r(0, y) = y^r$?

   (b) Show that $\varphi^r(x, y)$ may be expanded near the origin according to

   $$\varphi^a(x, y) = x^a + y^a + c^a{}_{bc}x^b y^c + O(x^2 y, xy^2). \tag{1}$$

   Use this to find $\bar{x}(x)$ for x small.

   (c) Let $g(d) = g(x)^{-1}g(y)^{-1}g(x)g(y)$ and show that for x, y small $d^a = f^a{}_{bc}x^b y^c$ where $f^a{}_{bc} = c^a{}_{bc} - c^a{}_{cb}$.

   (d) Using an expansion to one higher order show that the associativity condition $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ leads to the Jacobi identity.

   (e) Assume the Lie group has generators T_a satisfying $[T_a, T_b] = f^c{}_{ab}T_c$. For an element of the Lie algebra $a^a T_a$ there is an associated group element given by $g(a) = \exp(a^a T_a)$. Use the Baker–Campbell–Hausdorff formula $\exp tA \exp tB = \exp(t(A + B) + t^2[A, B]/2 + \mathcal{O}(t^3))$ to obtain $\varphi(x, y)$ for small x, y and verify that this is compatible with the general expansion of φ.

6. This question regards Pauli matrices. Verify the following properties of the Pauli matrices $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$:

   (a) $\sigma_i\sigma_j = I\delta_{ij} + i\epsilon_{ijk}\sigma_k$,

   (b) $\sigma_2\boldsymbol{\sigma}\sigma_2 = -\boldsymbol{\sigma}^*$,

   (c) $\exp(-i\theta\mathbf{n}\cdot\boldsymbol{\sigma}/2) = I\cos(\theta/2) - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)$.

7. Verify the Baker-Campbell-Hausdorff formula

   $$\exp tA\ \exp tB = \exp\left(t(A + B) + \frac{t^2}{2}[A, B] + \frac{t^3}{12}\{[A, [A, B]] + [B, [B, A]]\} + \dots\right)$$

   up to and including order t^2 (i.e. omitting the order t^3 term).

8. Let $g(t) = \exp it\sigma_1$. By evaluating $g(t)$ as a matrix, show that $\{g(t) : 0 \le t \le 2\pi\}$ is a 1-parameter subgroup of $SU(2)$. Describe geometrically how this subgroup sits inside the $SU(2)$ manifold.

9. Let $\exp iH = U$. Show that if H is Hermitian then U is unitary, and that if H is also traceless then $\det U = 1$. How do these results relate to the theorem that the exponential map $X \mapsto \exp X$ sends $L(G)$, the Lie algebra of G, to G?

10. A useful basis for the Lie algebra of $GL(n)$ consists of the n^2 matrices T^{ij} $(1 \le i, j \le n)$, where $(T^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Find the structure constants in this basis.

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