Examples Sheet 2

N.b. The general linear group of a vector space $GL(V)$ is the group of all automorphisms of V, i.e. bijective, linear maps $V \to V$. If V is finite dimensional and a basis is chosen, then $GL(V)$ is isomorphic to the general linear group of matrices $GL(\dim V, \mathbb{F})$.

1. (Warm-up) The dihedral group D_4 describes the symmetries of a square and is generated by a 90° rotation $r = R(\frac{\pi}{2})$ about its centre and a reflection m about the vertical (say) symmetry axis.

   (a) Write the group multiplication table for D_4.

   (b) Show that a representation of the dihedral group, $D : D_4 \to GL(2, \mathbb{R})$, can be constructed using the matrices

   $$D(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D(m) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Is this a faithful representation of D_4? Is it a reducible representation of D_4?

   (c) Consider the subgroup $K_4 = \{e, r^2, m, mr^2\}$ (Klein's *Vierergruppe*) and the corresponding matrices used above. Show that these four matrices constitute a reducible representation of K_4, and identify the invariant subspaces.

2. The adjoint representation of the Lie group $SU(2)$ is defined to be the map Ad : $SU(2) \to GL(\mathfrak{su}(2))$ given by:

   $$\mathrm{Ad}_A(X) = AXA^\dagger \qquad (*)$$

   for all $A \in SU(2), X \in \mathfrak{su}(2)$.

   (a) Show that Ad is indeed a group representation. This will require checking: (i) for each $A \in SU(2)$, we have that Ad_A is an automorphism of $\mathfrak{su}(2)$; (ii) given $A, B \in SU(2)$, we have $\mathrm{Ad}_{AB} = \mathrm{Ad}_A \circ \mathrm{Ad}_B$.

   (b) By writing $A = I + Y + O(Y^2)$ in $(*)$, construct the associated adjoint representation ad : $\mathfrak{su}(2) \to \mathfrak{gl}(\mathfrak{su}(2))$, where $\mathfrak{gl}(\mathfrak{su}(2))$ is the space of linear maps $\mathfrak{su}(2) \to \mathfrak{su}(2)$ of the Lie algebra $\mathfrak{su}(2)$. Verify that your proposed representation of $\mathfrak{su}(2)$ indeed constitutes a Lie algebra representation.

3. (a) If d_1 and d_2 are representations of a Lie algebra $L(G)$, show that $d_1 \oplus d_2$ is too. Via the exponential map, show that $\exp(d_1 \oplus d_2) = (D_1 \oplus D_2)(\exp)$ is a representation of G, where you may assume that D_i, where $D_i(\exp(X)) = \exp(d_i(X))$ for all $X \in L(G)$, constitute well-defined representations of the Lie group G for $i \in \{1, 2\}$.

   (b) Prove that the tensor product $d_1 \otimes d_2$ is a representation of $L(G)$. Exponentiate to show that $D_1 \otimes D_2$ is a representation of G.

4. Let D be a finite-dimensional representation of G acting on V, and $(\ ,\ )$ a positive definite inner product on V invariant under G, i.e.

$$(D(g)u, D(g)v) = (u, v) \ : \ u, v \in V\,,\ g \in G\,.$$

D is said to be unitary in this case.

(a) Let W be an invariant subspace of V. Show that $W_\perp$, the orthogonal complement of W in V, is also invariant.

(b) Deduce that D is completely reducible.

5. (Note that this question uses physics conventions for the generators t_i, such that they are Hermitian.) Three 3×3 matrices $\mathbf{t} := (t_1, t_2, t_3)$ are defined by $(t_i)_{jk} = -i\epsilon_{ijk}$.

(a) Prove $[t_i\,, t_j] = i\epsilon_{ijk}t_k$.

(b) Prove $(\mathbf{n}\cdot\mathbf{t})^3 = |\mathbf{n}|^2\mathbf{n}\cdot\mathbf{t}$.

(c) What are the possible eigenvalues of $\hat{\mathbf{n}}\cdot\mathbf{t}$ if $\hat{\mathbf{n}}$ is a unit vector?

(d) We may represent a rotation by an angle θ about an axis that points along the unit vector $\hat{\boldsymbol{n}}$ by the member of $SO(3)$ $R_{ij}(\hat{\mathbf{n}}, \theta) := \exp(-i\theta\hat{\mathbf{n}} \cdot \mathbf{t}\,)_{ij}$. By convention, $\hat{\mathbf{n}}$ points in any direction and $0 \le \theta \le \pi$. Evaluate R_{ij} explicitly by summing the Taylor series of the exponential, and show that

$$R_{ij}(\hat{\mathbf{n}}, \theta) = n_i n_j + (\delta_{ij} - n_i n_j)\cos\theta - \epsilon_{ijk}n_k \sin\theta\,.$$

(e) Verify the formula $e^{-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2}\,\sigma_j\,e^{i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2} = R_{ij}(\hat{\boldsymbol{n}}, \theta)\,\sigma_i\,.$

(f) Given an n-dimensional representation $D : G \to GL(n, \mathbb{C})$ of a group G, we can define its **conjugate representation** $\bar{D} : G \to GL(n, \mathbb{C})$ by complex conjugation: $\bar{D}(g) = D(g)^*$ for all $g \in G$. If D and $\bar{D}$ are inequivalent, then we say D is a **complex representation**. If D and $\bar{D}$ are equivalent, then there exists some invertible $n \times n$ matrix S such that $\bar{D}(g) = SD(g)S^{-1}$ for all $g \in G$. In this case, if $S^\mathsf{T} = S$, then D is said to be a **real representation**, otherwise $S^\mathsf{T} = -S$ and D is said to be **pseudoreal**. (These are the only two possibilities for equivalent, finite-dimensional representations.)

The set of matrices $\exp(-i\theta\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2)$ constitutes the defining representation of $G = SU(2)$. Show that this representation is pseudoreal and that the conjugate representation has the same weights as the original.

6. This question regards the explicit map of $SO(3) \cong SU(2)/\mathbb{Z}_2$.

(a) Show that $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. Why does this imply that any 2×2 matrix A can be expressed as

$$A = \frac{1}{2}\mathrm{Tr}(A)I + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}A) \cdot \boldsymbol{\sigma}?$$

(b) Define a one to one correspondence between real 3-vectors and Hermitian, traceless 2×2 matrices: $\boldsymbol{x} \to \boldsymbol{x} \cdot \boldsymbol{\sigma}$. Show that $\det(\boldsymbol{x} \cdot \boldsymbol{\sigma}) = -\boldsymbol{x}^2$.

(c) Next we define a transformation $\boldsymbol{x} \to \boldsymbol{x}'$ by $\boldsymbol{x}' \cdot \sigma = A\,\boldsymbol{x} \cdot \boldsymbol{\sigma} A^\dagger$, for $A \in SU(2)$. Deduce that $\boldsymbol{x}'^2 = \boldsymbol{x}^2$ and so $x_i' = R_{ij}x_j$ where $R \in SO(3)$. Finally, show

$$R_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_i A \sigma_j A^\dagger)\,.$$

(d) Show that $\sigma_j \sigma_i \sigma_j = -\sigma_i$ implies $\sigma_j A^\dagger \sigma_j = 2\mathrm{Tr}(A^\dagger)I - A^\dagger$ to obtain the equations $\sigma_i R_{ij} \sigma_j = 2\mathrm{Tr}(A^\dagger)A - I$ and $R_{jj} = |\mathrm{Tr}(A)|^2 - 1$.

(e) Why must $\mathrm{Tr}(A) \in \mathbb{R}$? Solve for $\mathrm{Tr}(A)$ and then A to show

$$A = \pm \frac{I + \sigma_i R_{ij} \sigma_j}{2\sqrt{1 + R_{jj}}}\,.$$

7. Finding the explicit map of $SO(1,3)^\uparrow \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ follows a similar calculation to the one finding the map of $SO(3) \cong SU(2)/\mathbb{Z}_2$ in Q6.

(a) Defining $\sigma_\mu = (I, \boldsymbol{\sigma}),\ \bar{\sigma}_\mu = (I,\ -\boldsymbol{\sigma})$, argue that any 2 by 2 matrix A may be written $A = \frac{1}{2}\mathrm{Tr}(\bar{\sigma}^\mu A)\sigma_\mu$.

(b) Now define a one-to-one correspondence between real 4-vectors x_μ and hermitian 2×2 matrices x, where $x_\mu \to \mathrm{x} = \sigma_\mu x^\mu$. Find $\det \mathrm{x}$ in terms of x_μ.

(c) For any $A \in SL(2,\mathbb{C})$, we define a linear transformation $\mathrm{x} \to_A \mathrm{x}' = A\mathrm{x}A^\dagger = x'^\dagger$. Show that $x^2 = x'^2$ and hence this must be a Lorentz transformation, so we can write $(x')^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\Lambda \in SO(1,3)^\uparrow$. Thus, show $\Lambda^\mu{}_\nu = \mathrm{Tr}(\bar{\sigma}^\mu A \sigma_\nu A^\dagger)/2$.

(d) To find the converse, show $\sigma_\nu A^\dagger \bar{\sigma}^\nu = 2\mathrm{Tr}(A^\dagger)I \Rightarrow \Lambda^\mu{}_\mu = |\mathrm{Tr}(A)|^2$ and $\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu = 2\mathrm{Tr}(A^\dagger)A$ and hence, for $\mathrm{Tr}(A) = e^{i\alpha}|\mathrm{Tr}(A)|$, $A = e^{i\alpha}\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu/(2\sqrt{\Lambda^\mu{}_\mu})$.

(e) Show that $\det A = 1$ determines $e^{i\alpha}$ up to a factor of ± 1. Thus $\pm A \leftrightarrow \Lambda$ ($\Lambda \in SO(1,3)^\uparrow$ because $SL(2,\mathbb{C})$ is continuously connected to the identity).

8. For a matrix Lie group G, consider the action of G on itself by conjugation, defined by $g' \to g g' g^{-1}$. Show that the eigenvalues of g' and $g g' g^{-1}$ are the same for all g, so the eigenvalues are invariants of an orbit.

Find the eigenvalues of the $SU(2)$ matrix $\cos\alpha/2\, I - i \sin\alpha/2\, \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\sigma}$ where $\boldsymbol{\alpha} = \alpha\hat{\boldsymbol{\alpha}}$. Deduce the orbit structure of $SU(2)$ under the action of $SU(2)$ on itself by conjugation.

9. (*Optional & nonexaminable extension question. Attempt only after finishing the questions above.*) Let V be the fundamental representation of $SO(3)$. Recall that a *rank r $SO(3)$-tensor* is an element of the tensor product representation

$$V^{\otimes r} := \underbrace{V \otimes V \otimes \ldots \otimes V}_{r \text{ times}}.$$

We define $V^{\otimes 0} := \mathbb{C}$ to be the trivial representation of $SO(3)$. If we pick a basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of V, then there is a natural basis $\{\vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_r} : i_1, \ldots, i_r = 1, 2, 3\}$ for the space $V^{\otimes r}$. In particular, given $T \in V^{\otimes r}$, we may write:

$$T = T_{i_1 i_2 \ldots i_r} \vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_r},$$

where $T_{i_1 i_2 \ldots i_r}$ are the *components* of the tensor with respect to this basis.

(a) Define a *transposition* $P_{(i,j)} : V^{\otimes r} \to V^{\otimes r}$ (with $1 \leqslant i < j \leqslant r$) of the space of rank-$r$ $SO(3)$-tensors by:

$$P_{(i,j)}(\vec{v}_1 \otimes ... \otimes \vec{v}_i \otimes ... \otimes \vec{v}_j \otimes ... \otimes \vec{v}_r) = \vec{v}_1 \otimes ... \otimes \vec{v}_j \otimes ... \otimes \vec{v}_i \otimes ... \otimes \vec{v}_r,$$

and the appropriate extension by linearity. Define a *trace* $T_{(i,j)} : V^{\otimes r} \to V^{\otimes (r-2)}$ (with $1 \leqslant i < j \leqslant r$) of the space of rank-$r$ $SO(3)$-tensors by:

$$T_{(i,j)}(\vec{v}_1 \otimes ... \otimes \vec{v}_i \otimes ... \otimes \vec{v}_j \otimes ... \otimes \vec{v}_r) = (\vec{v}_i \cdot \vec{v}_j)\, \vec{v}_1 \otimes ... \otimes \vec{v}_{i-1} \otimes \vec{v}_{i+1} \otimes ... \otimes \vec{v}_{j-1} \otimes \vec{v}_{j+1} \otimes ... \otimes \vec{v}_r,$$

and the appropriate extension by linearity. We say that a tensor $T \in V^{\otimes r}$ is *totally symmetric* if $P_{(i,j)}(T) = T$ for all $1 \leqslant i < j \leqslant r$, and we say that a tensor $T \in V^{\otimes r}$ is *totally traceless* if $T_{(i,j)}(T) = 0$ for all $1 \leqslant i < j \leqslant r$.

Show that a tensor $T \in V^{\otimes r}$ is totally symmetric and totally traceless if and only if its components with respect to some basis satisfy:

$$T_{(i_1 ... i_r)} = T_{i_1 ... i_r}, \qquad T_{kki_3 ... i_r} = 0.$$

(b) Let $W_r \subseteq V^{\otimes r}$ be the subset of totally symmetric, totally traceless tensors in $V^{\otimes r}$. Show that W_r is isomorphic to the $(2r+1)$-dimensional irreducible representation of $SO(3)$.

[Hint: First, show that W_r is an invariant subspace of $V^{\otimes r}$; therefore, it constitutes a valid representation of $SO(3)$. Next, apply the quadratic Casimir of the Lie algebra $\mathfrak{so}(3)$ to W_r and note its value. Finally, check dimensions to conclude.]

(c) Since $SO(3)$ is compact, $V^{\otimes r}$ is completely reducible. Let:

$$V^{\otimes r} = V_1 \oplus V_2 \oplus ... \oplus V_m$$

be a decomposition of $V^{\otimes r}$ into irreducibles (note that the decomposition may not be unique). By part (a), we know that for each $b = 1, ..., m$, there exists some a such that $V_b \cong W_a$. Let $\alpha : W_a \to V_b$ be an isomorphism of these two representations. Show that the components of the image $\alpha(S)$ are given by:

$$\alpha(S)_{j_1 ... j_r} = \alpha_{i_1 ... i_a j_1 ... j_r} S_{i_1 ... i_a},$$

where $\alpha_{i_1 ... i_a j_1 ... j_r}$ are the components of an $SO(3)$-invariant tensor.

(d) Hence, explain why the components T_{ij} of a general rank-2 $SO(3)$-tensor T may be decomposed as:

$$T_{ij} = \delta_{ij} S + \epsilon_{ijk} V_k + B_{ij} \qquad (*)$$

where $\delta_{ij} S$, $\epsilon_{ijk} V_k$, B_{ij} are the components of the projections of T onto irreducible subspaces of $V^{\otimes r}$, and B_{ij} is totally symmetric and totally traceless. By contracting $(*)$ with $SO(3)$ invariants, determine S, V_k and B_{ij} explicitly in terms of T_{ij}.

(e) Perform an analogous decomposition for the components of a rank-3 $SO(3)$-tensor, T_{ijk} (you should note in your construction that the decomposition is not in fact unique).

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