Symmetries, Fields and Particles. Examples 3

1. In this question and the following ones, $L(G)$ denotes the Lie algebra of a Lie group $G$ and $\mathbb{C}(G)$ denotes its complexification.
   
   i) Show that $\mathbb{C}(SU(2)) \simeq L(SL(2, \mathbb{C}))$

   where the RHS is considered as a complex Lie algebra.

   ii) By considering the subalgebra of real matrices show that $\mathbb{C}(SU(2))$ has two inequivalent real forms.

2. Write down a basis for a Cartan subalgebra of $\mathbb{C}(SO(2n))$ and for a Cartan subalgebra of $\mathbb{C}(SO(2n+1))$ for arbitrary integer $n$. Hence find the roots of the Lie algebras $\mathbb{C}(SO(3))$ and $\mathbb{C}(SO(4))$ and write down the corresponding step operators in your chosen basis.

3. If $\alpha$ is a root of a simple complex Lie algebra of finite dimension show that the only values of $k \in \mathbb{C}$ for which $k\alpha$ is a root are $k = \pm 1$.

4. Starting from the constraint, 
   $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \quad (*)$

   on the inner product of any two roots $\alpha$ and $\beta$, show that any complex simple Lie algebra of finite dimension has simple roots of at most two different lengths.

5. i) Using the constraint (*) of question 3, show that the off-diagonal elements of the Cartan matrix $A$ of a finite-dimensional complex simple Lie algebra of rank $r$ obey

   $A_{ij}A_{ji} < 4 \quad \text{for all} \quad i \neq j = 1, 2, \ldots, r.$

   ii) Consider possible $3 \times 3$ Cartan matrices of the form,

   $\begin{pmatrix} 2 & l & m \\
   l' & 2 & n \\
   m' & n' & 2 \end{pmatrix}$

   Using the constraints on the Cartan matrix derived in the lectures together with the result of part i), show that $l$, $m$ and $n$ cannot all be non-zero, and find all the allowed Cartan matrices with $m = 0$. Show that your solutions exhaust all the rank 3 simple Lie algebras in the Cartan classification. Why do no additional solutions with $m \neq 0$ arise?

6. Find a set of simple roots for the matrix Lie algebra $\mathbb{C}(SU(3))$ and determine the corresponding Cartan matrix showing that it coincides with the Cartan matrix of the Lie algebra $A_2$ in the Cartan classification.

7. Starting from its Dynkin diagram, construct the root system of the simple Lie algebra $B_2$.

8. The simple Lie algebra $A_2$ has simple roots $\alpha$, $\beta$ and a single additional positive root root $\theta = \alpha + \beta$. Choose a basis for the the Lie algebra consisting of the generators $\{h^\alpha, h^\beta, e^{\pm \alpha}, e^{\pm \beta}, e^{\pm \theta}\}$. Write down all brackets between Cartan generators and step operators and also the brackets between step operators including normalisation constants for any brackets which are potentially non-zero. Determine the constraints on the normalisation constants which follow from the Jacobi identity. How could you fix any remaining ambiguities?

9. Determine the weights of the $A_2$ representation with Dynkin labels $(2, 0)$. 