

Examples Sheet 4

1. A Lie algebra has simple roots $\alpha_1, \dots, \alpha_r$ and a diagonal Killing form. The fundamental weights satisfy

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

- (a) Show that $\alpha_i = \sum_j A_{ij} \omega_j$ where $[A_{ij}]$ is the Cartan matrix.
 - (b) A rank-2 Lie algebra has simple roots with coordinates $\alpha_1 = (1, 0)$ and $\alpha_2 = (-1, 1)$. What is the Cartan matrix?
 - (c) Assuming any other positive roots are equal in length to either one of the simple roots, show that $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$ are the other positive roots.
 - (d) Draw the root diagram, and show that the dimension of the Lie algebra is 10.
 - (e) Construct the fundamental weights ω_1, ω_2 .
 - (f) How is the highest weight of the representation whose weights coincide with the roots of the Lie algebra related to the fundamental weights?
2. Consider the Lie algebra with exactly 2 simple roots, $\alpha_1 = (1, 0)$ and $\alpha_2 = \frac{1}{2}(-3, \sqrt{3})$ (and a diagonal Killing form).
- (a) Determine the fundamental weights ω_1 and ω_2 . Let $|q_1, q_2\rangle$ be a state corresponding to the weight $q_1\omega_1 + q_2\omega_2$.
 - (b) Assuming $E_{\pm\alpha_i}, H_{\alpha_i}$ are the $SU(2)$ generators associated with the roots α_i construct a basis for the representation space starting from a highest weight vector $(i) |1, 0\rangle$ and $(ii) |0, 1\rangle$ by the successive action of $E_{-\alpha_1}$ and $E_{-\alpha_2}$ on the highest weight state.
 - (c) Show that the dimensions of the space are respectively 7 and 14 (in the second case there are two independent states with $q_1 = q_2 = 0$).
 - (d) Construct the weight diagram and in the 14-dimensional case show that it coincides with the root diagram.
3. A Lie algebra has a Cartan subalgebra $H = (H_1, \dots, H_r)$ and the remaining generators are E_α , corresponding to roots α , where $[H, E_\alpha] = \alpha E_\alpha$. Assume $[E_\alpha, E_{-\alpha}] = H_\alpha = 2(\alpha, H)/(\alpha, \alpha)$. For a root β , E_β satisfies

$$[E_\alpha, E_\beta] = 0, \quad [H_\alpha, E_\beta] = nE_\beta, \quad \underbrace{[E_{-\alpha}, [\dots, [E_{-\alpha}, E_\beta] \dots]]}_r = E_{\beta-r\alpha}.$$

- (a) Show that

$$[E_\alpha, E_{\beta-r\alpha}] = r(n - r + 1)E_{\beta-(r-1)\alpha}.$$

(b) Show that we may assume $E_{\beta-(n+1)\alpha} = 0$ for some integer n .

4. Decompose the following tensor product representations of $A_2 = \mathfrak{su}(3)$ into irreducible components: (a) $\mathbf{3} \otimes \bar{\mathbf{3}}$ and (b) $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. Discuss the connections between these irreducible representations and the quark model of light mesons and baryons [c.f. Wikipedia]. You might also think about the spin- $\frac{3}{2}$ baryons Δ^{++} and Ω^- , whose quark content is, respectively, uuu and sss , and what Pauli's exclusion principle implies about the quantum numbers of those quarks.
5. Find the smallest-dimension, irreducible representation of B_2 . Decompose the tensor product of two copies of this representation into irreps of B_2 , giving the dimension of each component.
6. Consider a gauge theory whose gauge group G is a matrix Lie group. The corresponding gauge field,

$$A_\mu : \mathbb{R}^{1,3} \rightarrow L(G),$$

transforms as

$$A_\mu \mapsto A'_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}$$

under a gauge transformation

$$g : \mathbb{R}^{1,3} \rightarrow G.$$

For the case $G = SU(N)$, check that $A'_\mu(x)$ takes values in the Lie algebra $L(G)$. Explain why this is true for any matrix Lie group G . Writing $g = \exp \varepsilon X$, with $\varepsilon \ll 1$, show that the corresponding infinitesimal gauge transformation is

$$A'_\mu = A_\mu + \varepsilon[X, A_\mu] - \varepsilon \partial_\mu X.$$

7. For a group with a Lie algebra with a basis $\{T_a\}$ such that $[T_a, T_b] = f^c_{ab} T_c$ let $\kappa_{ab} = (T_a, T_b)$ where $(\ , \)$ is an invariant symmetric bilinear form so that $([X, Y], Z) = -(Y, [X, Z])$.

(a) If D_μ is an appropriate covariant derivative involving a gauge field A_μ^a , verify

$$\partial_\mu(X(x), Y(x)) = (D_\mu X(x), Y(x)) + (X(x), D_\mu Y(x)).$$

(b) Let $T^\mu_\nu = (F^{\mu\sigma}, F_{\nu\sigma}) - \frac{1}{4}\delta^\mu_\nu(F^{\sigma\rho}, F_{\sigma\rho})$. Using the Bianchi identity, show that

$$\partial_\mu T^\mu_\nu = (D_\mu F^{\mu\sigma}, F_{\nu\sigma}).$$

(c) For a variation δA_μ^a obtain also $\delta \frac{1}{4}\epsilon^{\mu\nu\sigma\rho}(F_{\mu\nu}, F_{\sigma\rho}) = \partial_\mu \epsilon^{\mu\nu\sigma\rho}(\delta A_\nu, F_{\sigma\rho})$.

(d) By letting $A_\mu \rightarrow tA_\mu$, differentiating with respect to t , then integrating, show that

$$\frac{1}{4}\epsilon^{\mu\nu\sigma\rho}(F_{\mu\nu}, F_{\sigma\rho}) = \partial_\mu \epsilon^{\mu\nu\sigma\rho} \left(A_\nu, \partial_\sigma A_\rho + \frac{1}{3}[A_\sigma, A_\rho] \right).$$

8. With notation as in the previous question define a 3-dimensional Lagrangian

$$\mathcal{L} = \epsilon^{\mu\nu\rho} (\kappa_{ab} A_\mu^a \partial_\nu A_\rho^b + \frac{1}{3} f_{abc} A_\mu^a A_\nu^b A_\rho^c).$$

For a gauge transformation $\delta A_\mu^a = -\partial_\mu \lambda^a - f_{bc}^a A_\mu^b \lambda^c$ show that $\delta \mathcal{L} = -\partial_\mu (\epsilon^{\mu\nu\rho} \kappa_{ab} \lambda^a \partial_\nu A_\rho^b)$ so that $\int d^3x \mathcal{L}$ is invariant.

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