String Theory: Example Sheet 2

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1. Let z be a coordinate on the complex plane. Show that the vector fields

$$\ell_n = -z^{n+1}\partial_z \qquad n \in \mathbb{Z}$$

obey the Witt algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}\,.$$

Which of these vectors extend holomorphically to the Riemann sphere $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$? What subalgebra do they generate? *You may wish to consider the coordinate transformation $w = -z^{-1}$.*

2. Verify that

$$\partial\bar{\partial}\ln|z|^2 = 2\pi\delta(z,\bar{z})$$

firstly by using the divergence theorem, and secondly by regulating the singularity at $z = 0$.

3. The holomorphic stress tensor $T(z)$ has mode expansion $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$. Show that the modes L_n are given by the contour integral

$$L_n = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i}\, z^{n+1}\, T(z)\,.$$

Suppose that $\Phi(z,\bar{z})$ is a primary field of weight $(1,1)$ with mode expansion

$$\Phi(z,\bar{z}) = \sum_{n,\bar{n}} \phi_{n\bar{n}}\, z^{-n-1}\bar{z}^{-\bar{n}-1}\,,$$

where the modes $\phi_{n\bar{n}}$ obey $[L_m, \phi_{n\bar{n}}] = -n\,\phi_{n+m,\bar{n}}$ and $[\bar{L}_{\bar{m}}, \phi_{n\bar{n}}] = -\bar{n}\,\phi_{n,\bar{n}+\bar{m}}$. By considering the commutator of $\Phi(z,\bar{z})$ with the charge

$$Q = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i}\, v(w)T(w) + \oint_{\bar{w}=0} \frac{\mathrm{d}\bar{w}}{2\pi i}\, \bar{v}(\bar{w})\bar{T}(\bar{w})$$

for appropriately chosen $v(w)$, show that

i) Holomorphic translations $z \to z + c$ (with $c \in \mathbb{C}$) are generated by L_{-1},

ii) Dilations $z \to e^{\lambda}z \approx z + \lambda z$ (with $\lambda \in \mathbb{R}$) are generated by $L_0 + \bar{L}_0$,

iii) Rotations $z \to e^{i\theta}z \approx z + i\theta z$ (with $\theta \in \mathbb{R}/2\pi$) are generated by $L_0 - \bar{L}_0$.

[Hint: Start by choosing an appropriate form for $v(z)$ in each case (e.g. $v(z) = \lambda z$ in the case of dilations).]

4. Consider a theory of D free scalar fields $X^\mu(z,\bar{z})$ whose OPE is given by

$$X^\mu(z,\bar{z})\, X^\nu(w,\bar{w}) \;\sim\; \frac{\alpha'}{2}\,\eta^{\mu\nu}\,\ln|z-w|^2\,.$$

Suppose X^μ has a mode expansion with $\partial X^\mu(z) = -i\sqrt{\alpha'/2}\sum_n \alpha^\mu_n\, z^{-n-1}$. Show that the modes are recovered as

$$\alpha^\mu_n = i\sqrt{\frac{2}{\alpha'}}\oint \frac{\mathrm{d}z}{2\pi i}\, z^n\,\partial X^\mu(z)\,.$$

Hence show that the OPE implies the usual commutation relations $[\alpha^\mu_n,\alpha^\nu_m] = n\eta^{\mu\nu}\delta_{m+n,0}$ for these modes.

5. The holomorphic stress tensor of this free scalar theory is

$$T(w) = -\frac{1}{\alpha'}\;{:}\partial X^\mu\,\partial X_\mu{:}(w)$$

By considering the OPE of $X^\mu(z)$ with $T(w)$, prove that $\partial^n X^\mu(z)$ has definite conformal weight $(h,\bar{h}) = (n,0)$, but that it is a primary operator pnly when $n=1$.

6. Let $W(w)$ be the chiral operator

$$W(w) = \varepsilon_\mu\;{:}\partial X^\mu\, e^{ik\cdot X}{:}(w)$$

where k_μ and ε_μ are arbitrary constant space-time vectors. Show that $W(w)$ has conformal weight $h = 1+\alpha' k^2/4$. What condition(s) must we impose on (k_μ,ε_μ) if $W(w)$ is to be a primary field? Without doing any further calculations, give conditions on $\varepsilon_{\mu\nu}$ and k_μ for

$$V(w,\bar{w}) = \varepsilon_{\mu\nu}\;{:}\partial X^\mu\,\bar{\partial} X^\nu\, e^{ik\cdot X}{:}(w,\bar{w})$$

to be a primary field of weight $(1,1)$.

7. A free fermion Majorana fermion in two dimensions has action

$$S = \frac{1}{2\pi}\int \psi\,\bar{\partial}\psi + \bar{\psi}\,\partial\bar{\psi}\; d^2z\,,$$

so that the fields have OPE

$$\psi(z)\psi(w) \;=\; -\psi(w)\psi(z) \;\sim\; \frac{1}{z-w}$$

and similarly for $\bar{\psi}$. (Remember, ψ and $\bar{\psi}$ are Grassmann-valued fields, a fact which is reflected in the OPE). The energy momentum tensor is

$$T_{zz} = -\frac{1}{2}\;{:}\psi\,\partial\psi{:}$$

Show that ψ is a primary operator of weight $1/2$. Determine the central charge of this theory.

8. The bc ghost system consists of two free fermionic fields b and c. (Note: do not confuse the field c with the central charge c. They are not the same thing!) The OPE is given by

$$b(z)c(w) = -c(w)b(z) \;\sim\; \frac{1}{z-w}\,.$$

Consider the stress-energy tensor

$$T = :(\partial b)c: - \lambda\,\partial(:bc:)$$

where λ is a real constant. Show that b is primary with weight $h = \lambda$ and c is primary with weight $h = 1 - \lambda$. Show that the central charge of this system is equal to

$$\mathrm{c} = -12\lambda^2 + 12\lambda - 2$$

This peculiar looking theory is extremely important. We'll come across it later in the course when we discuss the path integral approach to string theory.