1. Given \( L_n = \frac{1}{2} \sum k \alpha_k \cdot \alpha_{-k} \), use the canonical commutation relations to show that \([L_n, \alpha_k] = -k \alpha_{k+n} \), and hence verify (without changing the operator ordering) that

\[
[L_m, L_n] = \frac{1}{2} \sum_{k \in \mathbb{Z}} [k \alpha_{n+k} \cdot \alpha_{m-k} + (m-k) \alpha_k \cdot \alpha_{m+n-k}] .
\]

Show that the substitution \( k \to k + m \) in the first term leads (apparently) to the result that \([L_m, L_n] = (m-n)L_{m+n} \). To analyse the \( n + m = 0 \) case, first split the sum over \( k \) as \( \sum_{k \in \mathbb{Z}} = \sum_{k<0} + \sum_{k=0} + \sum_{k>m} \). Next, assuming (without loss of generality) that \( m > 0 \), show that the sum over \( k < 0 \) does not contribute to \((0)[L_m, L_{-m}][0]\), where \([0]\) is the oscillator vacuum. Then use the identity

\[
k \alpha_{k-m} \cdot \alpha_{-k+m} + (m-k) \alpha_k \cdot \alpha_{-k} \equiv k \alpha_{-k+m} \cdot \alpha_{k-m} + (m-k) \alpha_{-k} \cdot \alpha_k
\]

to show that the sum over \( k > m \) does not contribute either. Hence show that

\[
\langle 0 | [L_m, L_{-m}] | 0 \rangle = m \alpha_0^2 + \frac{D}{2} \sum_{k=1}^{m-1} k(m-k).
\]

Use the identities \( \sum_{k=1}^{m-1} k = \frac{1}{2} m(m-1) \) and \( \sum_{k=1}^{m-1} k^2 = \frac{1}{6} m(m-1)(2m-1) \) to put this result in the form

\[
\langle 0 | [L_m, L_{-m}] | 0 \rangle = m \alpha_0^2 + \frac{D}{12} m^2(m-1).
\]

Deduce that the \( L_n \) span a Virasoro algebra with central charge \( c = D \).

2. In the “old covariant” approach to quantization of the open string with free ends, the general Lorentz-scalar state of momentum \( p \) at level 2 is \(|p\rangle \otimes |\phi\rangle\), where the oscillator Fock space state \(|\phi\rangle\) takes the form

\[
|\phi\rangle = \left[A \alpha_{-1}^2 + B \alpha_0 \cdot \alpha_{-2} + C (\alpha_0 \cdot \alpha_{-1})^2\right]|0\rangle,
\]

for constants \( A, B, C \), where \([0]\) is the Fock space vacuum. What is the value of \( \alpha_0^2 \) for this state? Show that \(|\phi\rangle\) satisfies the Virasoro conditions if \( 5B = (D - 1)A \) and \( 10C = (D + 4)A \), where \( D \) is the spacetime dimension. Now show, for \( A = 1 \), that

\[
\langle \phi | \phi \rangle = -(2/25) (D-1) (D-26).
\]

What can you conclude from this result?

3. Let \( A = \sum_n e^{i\sigma} A_n(t) \) be a function on the phase space of the closed NG string. Given that \( \delta_\xi A = -\sum_m \xi_m \{L_m, A\}_PB \) for parameter \( \xi = \sum_n e^{i\sigma} \xi_n(t) \), and

\[
\{L_m, A_n\}_PB = -i [m(h-1) - n] A_{n+m} ,
\]

show that \(-\delta_\xi A = \xi A' + h\xi' A\). Hence deduce that if \( A = A_- \) and \( \xi = \xi^- \), both depending only on \( \sigma^- \), then

\[
\delta_\xi A_- = \xi^- \partial_- A_- + h(\partial_- \xi^-) A_- .
\]

What is special about the \( h = 1 \) case?
4 The FP ghost action for the open string (with free ends) is

\[ I_{FP} = 2i \int dt \int_0^\pi d\sigma \left\{ b \partial_+ c + \bar{b} \partial_- \bar{c} \right\}. \]

Show that the boundary conditions on \((b, c)\) must be such that \(b \delta c = \bar{b} \delta \bar{c}\) at the endpoints. Verify that this condition is satisfied by the boundary conditions

\[ (\bar{b} - b)_{\text{ends}} = 0, \quad (\bar{c} - c)_{\text{ends}} = 0, \]

and verify that these boundary conditions are satisfied by the Fourier series expansions

\[ \bar{c}(-\sigma) = c(\sigma) = \sum_{n \in \mathbb{Z}} e^{in\sigma} c_n, \quad \bar{b}(-\sigma) = b(\sigma) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\sigma} b_n. \]

Hence show that the open string “quantum” action is

\[ I_{\text{qu}} = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} + \sum_{n \in \mathbb{Z}} ib_{-n} \dot{c}_n - H_{\text{qu}} \right\} \]

where \(H_{\text{qu}} = L_0 + N_{\text{gh}}\) with

\[ N_{\text{gh}} = \sum_{k=1}^{\infty} k (b_{-k} c_k + c_{-k} b_k) \]

5 Let \((b_k, c_k)\) be anticommuting variables with Poisson brackets \(\{b_{-k}, c_k\}_{PB} = -i\), and let

\[ L_m = \sum_{k \in \mathbb{Z}} [(J - 1)m - k] b_{m+k} c_{-k} \]

for some number \(J\). Show that

\[ \{L_m, c_n\}_{PB} = i (Jm + n) c_{n+m}, \quad \{L_m, b_n\}_{PB} = -i [(J - 1)m - n] b_{n+m}. \]

and hence that

\[ \{L_m, L_n\}_{PB} = -i (m - n) L_{m+n}. \]

Now quantise, and show that

\[ [L_m, L_{-m}] = \left( \sum_{k < 0}^{m} + \sum_{k=0}^{m} + \sum_{k > m}^{m} \right) (k - Jm)[k + (J - 1)m] (b_{k} c_{-k} - b_{-m+k} c_{m-k}). \]

Using the identity \(b_k c_{-k} - b_{-m+k} c_{m-k} = c_{m-k} b_{-m+k} - c_{-k} b_k\), and assuming that \(m > 0\), show that

\[ gh(0|L_m, L_n|0)_{gh} = -\frac{1}{6} m \left[ (6J^2 - 6J + 1) m^2 - 1 \right], \]

where \(|0\rangle_{gh}\) is annihilated by both \(b_k\) and \(c_k\) for \(k > 0\). Hence deduce that

\[ [L_m, L_{-m}] = 2m(L_0 - a) + \frac{c}{12} m(m^2 - 1), \quad a = \frac{1}{2} (J - 1), \quad c = -2 (6J^2 - 6J + 1). \]

Comment on the \(J = 2\) case. Why are \(a\) and \(c\) unchanged if \(J \rightarrow 1 - J\)? Comment on the \(J = 1/2\) case. What is \(c\) when \(J = 1/2\)?