String Theory: Example Sheet 3

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1. Show that an infinitesimal $\mathrm{SL}(2,\mathbb{C})$ transformation, acting as the Möbius group on $z \in \mathbb{C} \cup \infty$, takes the form

$$z \mapsto z' = \alpha + \beta z + \gamma z^2$$

where α, β and γ are constants. Hence show that the volume element of $\mathrm{SL}(2,\mathbb{C})$ may be written as

$$\mathrm{d}V_{\mathrm{SL}(2,\mathbb{C})} = \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2}$$

in terms the coordinates z_1, z_2, z_3 of any three distinct points on the Riemann sphere.

2. The ghost field $c(z)$ has mode expansion $c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}$. Show that the ghost vacuum $|0\rangle$ is defined by $c_n|0\rangle = 0$ for $n > 1$. Show that the bc CFT has correlation function

$$\langle c(z_1)c(z_2)c(z_3)\rangle_{bc} = K(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

for some constant K that you should find. *[You may assume that* $\langle 0|c_{-1}\, c_0\, c_1|0\rangle = 1$.*]* Hence show that

$$\prod_{i=1}^{n} \mathrm{d}^2 z_i \left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{bc,\bar{b}\bar{c}} \Big/ \ \mathrm{d}V_{\mathrm{SL}(2,\mathbb{C})} = \prod_{i=4}^{n} \mathrm{d}^2 z_i\,.$$

What is the significance of this result?

3. Show that the OPE between two plane wave operators $:e^{ik\cdot X}:$ takes the form

$$:e^{ik_1\cdot X}:(z_1,\bar{z}_1)\ \ :e^{ik_2\cdot X}:(z_2,\bar{z}_2)\ \sim\ |z_1 - z_2|^{\alpha' k_1\cdot k_2}\ :e^{ik_1\cdot X}(z_1,\bar{z}_1)\, e^{ik_2\cdot X}(z_2,\bar{z}_2):.$$

[You may find it useful to write the exponential operator as

$$:e^{ik\cdot X}:(z,\bar{z})\ =\ \sum_{n=0}^{\infty} \frac{i^n}{n!}\, k_{\mu_1}\cdots k_{\mu_n}\ :X^{\mu_1}\cdots X^{\mu_n}:(z,\bar{z})$$

and first consider the OPE between $X^\mu(z_1,\bar{z}_1)$ *and* $:e^{ik\cdot X}:(z_2,\bar{z}_2)$.*]*

4a. The scattering amplitude for m closed string tachyons is given by

$$\mathcal{A}^{(m)}(k_1,\ldots,k_m) = \frac{g_s^{m-2}}{\mathrm{Vol}(\mathrm{SL}(2,\mathbb{C}))} \int \prod_{i=1}^{m} d^2 z_i\ \langle \hat{V}(z_1,k_1)\ldots\hat{V}(z_m,k_m)\rangle\,,$$

where $\hat{V}(z,k) = e^{ik\cdot X(z,\bar{z})}$ and the correlation function is computed using the gauge fixed free Polyakov action

$$S_{\mathrm{Poly}} = \frac{1}{2\pi\alpha'} \int \partial X^\mu\, \bar{\partial} X_\mu\, d^2 z\,.$$

By expressing the correlation function as a Gaussian integral, show that the amplitude is

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\text{Vol}(\text{SL}(2,\mathbb{C}))}\,\delta^{26}\left(\sum_i k_i\right)\int\prod_{i=1}^{m} d^2z_i\,\prod_{j<l}|z_j - z_l|^{\alpha' k_j\cdot k_l}\,.$$

b. Show that this integral is invariant under the worldsheet SL$(2,\mathbb{C})$ transformation

$$z_i \to \frac{az_i + b}{cz_i + d}$$

only when the momenta are on-shell, *i.e.* $k_i^2 = 4/\alpha'$ for each i. Hence explain why the 4-tachyon amplitude can be reduced to the integral

$$\mathcal{A}^{(4)} \sim g_s^2\,\delta^{26}\left(\sum_i k_i\right)\int d^2z\;|z|^{\alpha' k_2\cdot k_3}\,|1-z|^{\alpha' k_3\cdot k_4} \qquad (\dagger)$$

c. By evaluating the remaining integral in terms of Γ-functions, show that string theory reproduces the Virasoro-Shapiro amplitude

$$\mathcal{A}^{(4)} \sim g_s^2\,\delta^{26}\left(\sum_i k_i\right)\frac{\Gamma(-1-\alpha' s/4)\,\Gamma(-1-\alpha' t/4)\,\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\,\Gamma(2+\alpha' t/4)\,\Gamma(2+\alpha' t/4)}\,,$$

where s, t, u are the usual Mandelstam variables.

5. Explain why the limit $s\to\infty$, with t fixed corresponds to small angle scattering at high energy. Show that in this limit the Virasoro-Shapiro amplitude exhibits so-called Regge behaviour,

$$\mathcal{A}^{(4)} \to\; g_s^2\,\delta^{26}(\sum_i p_i)\,\frac{\Gamma(-1-\alpha' t/4)}{\Gamma(2+\alpha' t/4)}\,s^{2+\alpha' t/2}\,.$$

Show that this result agrees with the saddle point approximation to the integral ($\dagger$) above.

6a. Write down the vertex operator for a massless closed string state with polarization $\zeta_{\mu\nu}$ and momentum k_μ. What are the restrictions on k_μ and $\zeta_{\mu\nu}$?

b. Consider the scattering of a massless closed string mode with momentum k_1 and two tachyons with momenta k_2 and k_3. Show that $k_1\cdot k_2 = k_1\cdot k_3 = 0$, while $k_2\cdot k_3 = -4/\alpha'$.

c. Show that the 3-point scattering amplitude for these particles is given by

$$\mathcal{A}^{(3)} \sim \frac{g_s}{\text{Vol}(\text{SL}(2,\mathbb{C}))}\,\delta^{26}\left(\sum_i k_i\right)\int\prod_{i=1}^{3} d^2z_i\,\frac{1}{|z_{23}|^4}\,\zeta_{\mu\nu}\left(\frac{k_2^\mu}{z_{12}}+\frac{k_3^\mu}{z_{13}}\right)\left(\frac{k_2^\nu}{\bar z_{12}}+\frac{k_3^\nu}{\bar z_{13}}\right)$$

where $z_{ij} = z_i - z_j$.

d. Show that the residual SL$(2;\mathbb{C})$ gauge redundancy allows us to simplify this to

$$\mathcal{A}^{(3)} \sim g_s\,\delta^{(26)}\left(\sum_i k_i\right)\;\zeta_{\mu\nu}(k_2^\mu - k_3^\mu)(k_2^\nu - k_3^\nu)\,. \qquad (\ddagger)$$

e. Consider the 26-dimensional theory

$$S[G,\Phi] = -\frac{1}{2\kappa^2}\int \left(R(G) + \frac{1}{2}G^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi + \frac{1}{2}M^2\Phi^2 \right)\sqrt{-G}\,d^{26}x$$

of gravity coupled to a real scalar field Φ of mass $M^2 = -4/\alpha'$. Show that the tree-level amplitude for a single graviton of polarization $\zeta_{\mu\nu}$ and momentum k_1 to be emitted from a scalar of momentum k_2 agrees with the result (‡). Why does this amplitude not receive stringy corrections?