String Theory: Example Sheet 4

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1a. The 1-loop worldsheet β-functions for the closed string are

$$\begin{aligned}\beta_{\mu\nu}(G) &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} \\ \beta_{\mu\nu}(B) &= -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi\, H_{\lambda\mu\nu} \\ \beta(\Phi) &= -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi\, \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}\,.\end{aligned}$$

Show that the conditions $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$ are equivalent to the equations of motion for the effective action

$$S[G,B,\Phi] = \frac{1}{2\kappa_0^2} \int \left(R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi\, \partial^\mu \Phi \right) e^{-2\Phi} \sqrt{-G}\, d^{26}X \qquad (\dagger)$$

in the target space, written in the string frame.

b. The worldsheet β-functions receive an infinite series of higher loop corrections. Without detailed calculation, briefly discuss how these affect the target space effective action.

2. Consider the string frame action ($\dagger$), but in D space-time dimensions. Show that, when written in terms of the Einstein frame metric

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)}\, G_{\mu\nu}(X)\,,$$

the low-energy effective action becomes

$$S = \frac{1}{2\kappa^2} \int d^D X \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{12} e^{-8\tilde{\Phi}/D-2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{4}{D-2} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} \right)\,,$$

where $\kappa^2 = \kappa_0^2\, e^{2\Phi_0}$ and $\Phi = \Phi_0 + \tilde{\Phi}$.

3a. The string frame metric produced by N infinite static strings lying in the $(X^0, X^1) \equiv (t,x)$ direction is

$$ds^2 = f(r)^{-1}(-dt^2 + dx^2) + d\vec{X} \cdot d\vec{X}\,,$$

where $\vec{X} = (X_2, \ldots, X_{25})$ labels the space transverse to the string and

$$f(r) = 1 + g_s^2 N \left(\frac{l_s}{r} \right)^{22} \qquad \text{with} \qquad r^2 = \vec{X} \cdot \vec{X}\,.$$

Consider one further infinite probe string in this background, lying parallel to the others. Write down the Nambu-Goto action describing the motion of this string. Show that in static gauge $t = R\tau$ and $x = R\sigma$, the low-energy excitations of the string are governed by the effective action

$$L \approx T \int \left[-f(r)^{-1} + \frac{1}{2}\left(\frac{d\vec{X}}{dt} \cdot \frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx} \cdot \frac{d\vec{X}}{dx} \right) + \dots \right] dt\, dx\,.$$

Interpret this result.

b. The N strings also source a background B-field, given by

$$B_{01} = f(r)^{-1} - 1\,.$$

Including the coupling its to this background B-field, show that the probe string, suitably oriented and lying parallel to the initial strings, feels no static force.

4. Consider an open string whose ends are constrained to lie on a Dp-brane with a background field-strength F_{ab} turned on. Show that the Neumann boundary conditions for the string must be replaced by

$$\left(\partial_\sigma X^a - 2\pi\alpha'\, F^{ab}\, \partial_\tau X_b\right)\Big|_{\sigma=0,\pi} = 0\,.$$

5a. The Born-Infeld Lagrangian is defined to be $\mathcal{L}_{BI} \equiv \sqrt{\det(\mathbb{1} + F)}$, where $\mathbb{1}$ is the $D \times D$ identity matrix, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field-strength and we have set $2\pi\alpha' = 1$. Show that this Lagrangian may also be written in the form

$$\mathcal{L}_{BI} = \exp\left(\frac{1}{4}\text{tr}\,\ln(\mathbb{1} - F^2) \right)\,.$$

where F^2 represents the matrix $F^{\mu\kappa}F_{\kappa\nu}$.

b. Use the Bianchi identity to show that

$$\partial_\mu \left[\text{tr}\,\ln(\mathbb{1} - F^2)\right] = -4\,\partial_\rho F_{\mu\sigma} \left(\frac{F}{\mathbb{1} - F^2} \right)^{\sigma\rho}\,.$$

and also that

$$\partial_\mu \left(\frac{F}{\mathbb{1} - F^2} \right)^{\mu\nu} = \left(\frac{F}{\mathbb{1} - F^2} \right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left(\frac{F}{\mathbb{1} - F^2} \right)^{\sigma\nu} + \left(\frac{1}{\mathbb{1} - F^2} \right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left(\frac{1}{\mathbb{1} - F^2} \right)^{\sigma\nu}$$

c. The 1-loop worldsheet β-functions for the open string include

$$\beta_\sigma(F) = \left(\frac{1}{\mathbb{1} - F^2} \right)^{\mu\rho} \partial_\mu F_{\rho\sigma}\,.$$

Show that the conditions $\beta_\sigma(F) = 0$ are equivalent to the equations of motion arising from the Born-Infeld action.

6. Consider open strings in D-dimensional Minkowski space with endpoints that satisfy

Dirichlet boundary conditions in the directions $X^{p+1}, X^{p+2}, \ldots, X^{25}$ and Neumann conditions in the remaining directions.

Write down the classical mode expansion for an open string suspended between two separated, parallel Dp-branes.

Calculate the quantum ground state energy for this string. Find the critical seperation at which this energy vanishes.

Show that the 1st excited state of the string contains a "W-boson", a massive spin 1 particle charged under the two $U(1)$ gauge fields living on the branes. Show that this state becomes massless as the brane coincide in spacetime. Why does this mean a non-Abelian $U(2)$ gauge symmetry emerges? Now consider this system when the direction X^{25} is a circle of radius R. What is the dual description of this system that is obtained by a T-duality transformation?

7a. A closed string moves through a space-time $\mathbb{R}^{24,1} \times S^1$ where one dimension is compactified on a circle of radius R. We write the worldsheet fields as

$$(X^i, Y) : \Sigma \to \mathbb{R}^{24,1} \times S^1,$$

so that the coordinate along the circle is periodic; $Y \sim Y + 2\pi R$. Explain why the periodicity condition on the string is

$$Y(\sigma + 2\pi, \tau) = Y(\sigma, \tau) + 2\pi R w\,,$$

taking care to give an interpretation to $w \in \mathbb{Z}$. Hence find a mode expansion for $Y(\sigma, \tau)$.

b. Show that the mass-shell and level-matching conditions are

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \bar{N} - 2) \qquad \text{and} \qquad nw + N - \bar{N} = 0,$$

respectively, where $n, w \in \mathbb{Z}$. N and $\bar{N}$ take their usual meaning. Hence show that the mass spectrum of the theory is invariant under the exchange of winding and momentum modes around the compact direction provided that R is exchanged with α'/R. What is the significance of this result? *[Hint: Do not assume that $\alpha_0^\mu = \bar{\alpha}_0^\mu$.]*

c. Show that, when the radius R is generic, the massless states have level $N = \tilde{N} = 1$, with winding $w = 0$ and Kaluza-Klein momentum $n = 0$. Show that, from the perspective of an observer in $\mathbb{R}^{1,24}$, these states are a graviton, an anti-symmetric tensor, two scalars and two $U(1)$ vector fields.

d. Consider the limit in which $R \to 0$, keeping α' fixed. How do the low energy degrees of freedom for the resulting 25 dimensional theory on $\mathbb{R}^{24,1}$ in string theory differ from those in field theory?

8a. As in the previous question, assume that the target space is $\mathbb{R}^{24,1} \times S^1$, but where the radius of the circle is now taken to be the special value $R = \sqrt{\alpha'}$. Show that there are extra massless states with $N = \tilde{N} = 0$, and with $N = 1$, $\tilde{N} = 0$. What is the interpretation of these states?

b. Using the standard OPE for the free scalar Y, find the singular terms in the OPEs between the operators

$$J^1(z) \;=\, :\sin\left(\frac{2}{\sqrt{\alpha'}}Y(z)\right):\,, \qquad J^2(z) \;=\, :\cos\left(\frac{2}{\sqrt{\alpha'}}Y(z)\right): \qquad \text{and} \qquad J^3(z) \;=\; \frac{i}{\sqrt{\alpha'}}\,\partial Y(z)\,.$$

Hence show that the modes

$$J^a_n = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^n \, J^a(z)\,,$$

obey the commutation relations

$$[J^a_m, J^b_n] = \frac{m}{2}\delta^{ab}\,\delta_{m+n,0} + i\epsilon^{abc}J^c_{m+n}.$$

[Such an algebra is called a Kac-Moody algebra, or an affine Lie algebra. It plays a prominent role in the worldsheet of the heterotic string.]