Supersymmetry and Extra Dimensions:
Example Sheet 4

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Exercise 4.1: Consider the Schrödinger equation for a particle moving in two dimensions $x$ and $y$. The second dimension is a circle or radius $r$. The potential corresponds to a square well ($V(x) = 0$ for $x \in (0, a)$ and $V = \infty$ otherwise). Derive the energy levels for the two-dimensional Schrödinger equation and compare the result with the standard one-dimensional situation in the limit $r \ll a$.

Exercise 4.2: Consider the following Lagrangian

$$S = \int d^4x \left( \frac{1}{g^2} H_{\mu\nu\rho} H^{\mu\nu\rho} + \alpha \epsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma} \right).$$

Solve the equation of motion for the Lagrange multiplier $\alpha$ to obtain an action for a propagating massless Kalb-Ramond field $B_{\mu\nu}$. Alternatively, solve the equation of motion for the field $H_{\nu\rho\sigma}$, to obtain an action for the propagating axion field $\alpha$. What happens to the coupling $g$ under this transformation? Generalise your result for arbitrary dimensions and ranks of the tensors.

Exercise 4.3: Consider a massive antisymmetric tensor of rank $q$ in $D$ dimensions. Write up its Lagrangian up-to second derivatives. Describe a general Lagrangian that can reproduce the original Lagrangian and its dual. Determine the degrees of freedom of the original and dual tensors. Interpret this dualisation in terms of a functional Fourier transform. Can this also be used in the massless case?

Exercise 4.4 On spacetimes with Lorentzian signature show that only in dimensions $D = 4k + 2$ there can be self-dual antisymmetric tensors. How many degrees of freedom do they have? What kind of $p$-branes they couple to? Explain the difference, if any, with Euclidean spaces.

Exercise 4.5: Show that the Kaluza-Klein dimensional reduction from $D = 5$ to $D = 4$ follows from a pure gravitational theory in five-dimensions, using $(\nabla)R = (\nabla)R - 2e^{-\sigma} \nabla^2 e^{\sigma} - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu}$ where $G_{55} = e^{2\sigma}$. Relate the gauge transformation to the $U(1)$ isometry of the compact space.

Exercise 4.6 Demonstrate that the volume of a $N - 1$ sphere of radius $r$ is

$$V_{N-1} = \frac{2 \pi^{N/2}}{\Gamma(N/2)} r^{N-1} \quad (1)$$

Hint: It may help to consider the integral $I_N = \int d^N x e^{-\rho^2}$ with $\rho^2 = \sum_{i=1}^N x_i^2$. Use this result to derive an expression for the electric (and gravitational) potential in $D$ dimensions. Show that the potential due to a point particle in five dimensions reduces to the 4-dimensional potential at distances much larger than the size of the fifth dimension.
Exercise 4.7: Consider a five dimensional gravity theory with a negative cosmological constant $\Lambda < 0$, compactified on an interval $(0, \pi)$. Each end of the interval corresponds to a ‘3-brane’ which we choose to have tension $\pm \Lambda/k$ respectively. Here $k$ is a common scale to be determined later in terms of the fundamental scale in 5D $M_*$ and $\Lambda$. Verify that the warped metric

$$ds^2 = e^{-2A(\theta)} \eta_{\mu\nu} dx^\mu dx^\nu - r^2 d\theta^2$$

satisfies Einstein’s equations. Here $W = e^{-2A(\theta)}$ is the warp factor and $r$ is a constant measuring the size of the interval. You can use that Einstein’s equations reduce to

$$\frac{6 A'^2}{r^2} = -\frac{\Lambda}{2M_*^4}, \quad \frac{3 A''}{r^2} = \frac{\Lambda}{2M_*^4 kr} \left[ \delta(\theta - \pi) - \delta(\theta) \right].$$

Solve for $A(\theta)$ and use the warp factor to show that the effective 4D Planck scale is now

$$M_{\text{pl}}^2 = M_*^3 r \int_{-\pi}^{\pi} d\theta e^{-2A} = \frac{M_*^3}{k} (1 - e^{-2kr}).$$

Find the value of the constant $k$. Consider the Higgs Lagrangian on the brane at $\theta = \pi$, bring it into canonical form and show that the mass is proportional to the factor $e^{-k\pi r}$. How large can $r$ be in order to reproduce the electroweak scale from the Planck scale? Does this solve the hierarchy problem? How does the Planck scale differ from the 5D scale $M_*$?

Exercise 4.8: Imagine that it were possible to have particles with all possible spins up to $j = 3$. What would the maximum dimensionality of spacetime be consistent with supersymmetry?

Exercise 4.9: Starting with the field contents of IIA and IIB supergravities in $D = 10$ perform the dimensional reduction to $D = 9$ and count the number of degrees of freedom for each multiplet. Is the spectrum chiral? Perform directly the reduction from $D = 11$ to $D = 9$ and compare. Perform dimensional reduction of IIB supergravity in $D = 10$ all the way to $D = 4$ and compare the number of degrees of freedom.

Exercise 4.10: Consider $\mathcal{N} = 1$ supergravity with three chiral superfields $S, T,$ and $C$. In Planck units, the Kähler potential and superpotential are given by

$$K = -\log (S + S^*) - 3 \log (T + T^* - CC^*)$$

$$W = C^3 + a e^{-aS} + b,$$

where $a, b$ are arbitrary complex numbers and $a > 0$. Compute the scalar potential. Find the auxiliary field for $S, T, C$ and verify that supersymmetry is broken. Assuming that $C$ denotes a matter field with vanishing vev, find a minimum of the potential. Are there flat directions? A typical Kähler potential derived from string compactifications takes the form

$$K = -3 \log \Gamma(\tau_i)$$

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where $\Gamma$ is a homogeneous function of degree one of moduli fields $\tau_i$. Using the homogeneity equations $\tau_i \Gamma_i = \Gamma$ and $\tau_i \Gamma_{ij} = 0$ (where $\Gamma_i = \partial \Gamma/\partial \tau_i$, etc.) show that

$$\tau_i K_{ij} = 3 \Gamma_j / \Gamma, \quad \Gamma_i K^{-1}_{ij} \Gamma_j / \Gamma^2 = 1/3$$

and deduce from this that if the superpotential does not depend on the $\tau_i$ fields then the corresponding contribution to the $\mathcal{N} = 1$ supergravity scalar potential $V$ vanishes.