General Relativity: Example Sheet 1

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1. Draw the integral curves corresponding to a vector field in $\mathbf{R}^2$ with components, in Cartesian coordinates, given by

- $X^\mu = (y, -x)$
- $X^\mu = (x - y, x + y)$

[Hint: you might find life easier in polar coordinates.]

2. Let $\hat{H} : T_p(M) \to T_p^*(M)$ be a linear map. Define

$$H(X, Y) = [\hat{H}(Y)](X)$$

Show that this map is linear in both arguments (e.g. $H(fX + gY, Z) = fH(X, Z) + gH(Y, Z)$ for $f, g \in \mathbf{C}$ and $X, Y, Z \in T_p(M)$) and hence defines a rank $(0, 2)$ tensor.

Similarly, show that a linear map $T_p(M) \to T_p(M)$ defines a tensor of rank $(1, 1)$. What tensor δ arises from the identity map?

3. Let $V^{\mu\nu}$ be the components an arbitrary rank $(2, 0)$ tensor, and $S_{\mu\nu}$ and $A_{\mu\nu}$ be the components of symmetric and anti-symmetric rank $(0, 2)$ tensors respectively (i.e. $S_{\mu\nu} = S_{\nu\mu}$ and $A_{\mu\nu} = -A_{\nu\mu}$). Show that $V^{\mu\nu}S_{\mu\nu} = V^{(\mu\nu)}S_{\mu\nu}$ and $V^{\mu\nu}A_{\mu\nu} = V^{[\mu\nu]}A_{\mu\nu}$.

4. You are given a rank $(2, 0)$ tensor K. Working first in some basis, devise a criterion to test whether it is the *direct product* of two vectors A, B, i.e., $K^{\mu\nu} = A^\mu B^\nu$.

Can you express the test in a manifestly basis-invariant manner? [Hint: one option is to use determinants, but it is not the only one.]

Show that the general rank $(2, 0)$ tensor in n dimensions cannot be written as a direct product, but can be expressed as a sum of many direct products.

5*. Let M be a manifold and $f : M \to R$ be a smooth function such that $\mathrm{d}f = 0$ at some point $p \in M$. Let x^μ be a coordinate chart defined in a neighbourhood of p. Define

$$F_{\mu\nu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} .$$

By considering the transformation law for components show that $F_{\mu\nu}$ defines a rank $(0, 2)$ tensor. (This is called the *Hessian* of f at p.) Construct also a coordinate-free definition and demonstrate its tensorial properties.

6. Let $g_{\mu\nu}$ be a rank $(0,2)$ tensor. In a basis, one can regard the components $g_{\mu\nu}$ as elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{\mu\nu})$. How does g transform under a change of basis?

7*. Use the Leibniz rule to derive the formula for the Lie derivative of a 1-form ω, valid in any coordinate basis:

$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$$

[Hint: consider $(\mathcal{L}_X \omega)(Y)$ for a vector field Y.] Show that the Lie derivative of a $(0,2)$ tensor g is

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho$$

For a p-form η, define $\iota_X \eta$ to be the $(p-1)$-form that results from contracting a vector field X with the first index of η. Show that for a 1-form ω,

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega)$$

8. Let ω be a p-form and η a q-form. Show that the exterior derivative satisfies the properties

- $d(d\omega) = 0$
- $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta$
- $d(\varphi^* \omega) = \varphi^*(d\omega)$ where $\varphi : M \to N$ for some manifolds M and N

9. The exterior derivative of $\omega \in \Lambda^p(M)$ can be defined as

$$\begin{aligned} d\omega(X_1, \ldots, X_{p+1}) \;=\; & \sum_{j=1}^{p+1} (-1)^{j-1} X_j(\omega(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{p+1})) \\ & + \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{k-1}, X_{k+1}, \ldots X_{p+1}) \end{aligned}$$

In a coordinate basis $\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$, use this definition to determine the components of $d\omega$ in the cases $p = 1$ and $p = 2$.

10. A three-sphere can be parameterized by Euler angles (θ, ϕ, ψ) where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. Define the following 1-forms

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi\ , \quad \sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi\ , \quad \sigma_3 = d\psi + \cos\theta\, d\phi$$

Show that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ with analogous results for $d\sigma_2$ and $d\sigma_3$.

11. Let $\{e_\mu\}$ be a basis of vectors fields, with

$$[e_\mu, e_\nu] = \gamma^\rho{}_{\mu\nu}\, e_\rho.$$

The functions $\gamma^\rho{}_{\mu\nu}$ are known as *commutator components.* For a choice of coordinates $\{x^\mu\}$, we often work with the *coordinate induced* basis $e_\mu = \{\partial_\mu\}$. Show that, in this case, $[e_\mu, e_\nu] = 0$.

The purpose of this question is to show the converse: that $[e_\mu, e_\nu] = 0$ only for a coordinate induced basis. Consider a general basis $\{e_\mu\}$ and the dual basis $\{f^\mu\}$ of one-forms. In general, these can be expanded as

$$e_\mu = e_\mu{}^\rho \frac{\partial}{\partial x^\rho} \quad \text{and} \quad f^\mu = f^\mu{}_\rho dx^\rho$$

where $e_\mu{}^\rho f^\nu{}_\rho = \delta_\mu{}^\nu$. Show that

$$e_\mu{}^\sigma \frac{\partial e_\nu{}^\lambda}{\partial x^\sigma} - e_\nu{}^\sigma \frac{\partial e_\mu{}^\lambda}{\partial x^\sigma} = \gamma^\rho{}_{\mu\nu}\, e_\rho{}^\lambda$$

Hence deduce that

$$e_\mu{}^\sigma e_\nu{}^\lambda \frac{\partial f^\rho{}_\lambda}{\partial x^\sigma} - e_\nu{}^\sigma e_\mu{}^\lambda \frac{\partial f^\rho{}_\lambda}{\partial x^\sigma} = -\gamma^\rho{}_{\mu\nu}\ ,$$

and finally that

$$\frac{\partial f^\rho{}_\sigma}{\partial x^\lambda} - \frac{\partial f^\rho{}_\lambda}{\partial x^\sigma} = -\gamma^\rho{}_{\mu\nu} f^\mu{}_\lambda f^\nu{}_\sigma$$

Use this result, together with the Poincaré lemma, to show that if $[e_\mu, e_\nu] = 0\ \forall\ \mu, \nu$ then the basis is coordinate induced.