

# General Relativity: Example Sheet 4

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1. In almost-inertial coordinates, the energy-momentum tensor describing a stationary point mass at the origin is

$$T_{00}(t, \mathbf{x}) = M\delta^3(\mathbf{x})$$

with  $T_{0i} = T_{ij} = 0$ . Determine the linearized gravitational field produced by this energy-momentum tensor, assuming it to be independent of  $t$ . For what values of  $R = |\mathbf{x}|$  is the linear approximation valid?

2\*. In almost inertial coordinates the energy momentum tensor of a straight *cosmic string* aligned along the  $z$ -axis is

$$T_{\mu\nu} = \mu \delta(x)\delta(y) \text{diag}(1, 0, 0, -1),$$

where  $\mu > 0$  is constant. Show that the linearised Einstein equations admit a solution with non-vanishing metric perturbation components

$$h_{11} = h_{22} = -\lambda \quad \text{with} \quad \lambda = 8\mu G \log\left(\frac{r}{r_0}\right)$$

where  $r = \sqrt{x^2 + y^2}$  and  $r_0$  is an arbitrary length.

Show that the perturbed metric can be written in cylindrical polar coordinates as

$$ds^2 = -dt^2 + dz^2 + (1 - \lambda)(dr^2 + r^2 d\phi^2).$$

Working to leading order in  $\mu G$ , make a change of radial coordinate given by  $(1 - \lambda)r^2 = (1 - 8\mu G)\tilde{r}^2$  to obtain

$$ds^2 = -dt^2 + dz^2 + d\tilde{r}^2 + (1 - 8\mu G)\tilde{r}^2 d\phi^2,$$

Finally, change the angular coordinate to obtain

$$ds^2 = -dt^2 + dz^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2.$$

Is this Minkowski spacetime? Show intuitively how a distant object behind a cosmic string may appear as a double image.

3. Consider a large thin spherical shell of mass  $M$  and radius  $R$  which rotates slowly about the  $z$ -axis with angular velocity  $\Omega$ . The energy density is

$$\rho = \frac{M}{4\pi R^2} \delta(r - R)$$

and the 4-velocity is  $u^\mu = (1, -\Omega y, \Omega x, 0)$ .

i.) Solve the linearised Einstein equations sourced by  $T_{00}$ . Show that the result agrees with Newtonian theory.

ii.) Solve the linearised Einstein equations sourced by  $T_{0i}$ . Show that the solution is given by

$$h_{0i} = \begin{cases} \omega(y, -x, 0) & r < R \\ \frac{\omega R^3}{r^3}(y, -x, 0) & r > R \end{cases}$$

where  $\omega = 4MG\Omega/3R$ .

[Hint: Introduce the complex combination  $H = h_{01} + ih_{02}$  and work in spherical polar coordinates. You will need the Laplacian,

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)$$

Look for a solution of the form  $H = f(r) \sin \theta e^{i\phi}$ .]

iii.) Consider a free particle, moving inside the shell. For slowly moving particles, we may take the 4-velocity to be  $v^\mu = (1, \dot{\mathbf{x}})$  with  $|\dot{\mathbf{x}}| \ll 1$ . Working to leading order in both  $|\dot{\mathbf{x}}|$  and  $\omega$ , show the geodesic equation implies

$$\ddot{\mathbf{x}} = 2\boldsymbol{\omega} \times \dot{\mathbf{x}}$$

where  $\boldsymbol{\omega} = (0, 0, \omega)$ .

[Note: this is the equation that we usually associate to a rotating frame, where the right-hand-side is identified as the Coriolis force. In general relativity, this phenomenon is known as *frame-dragging*, or the *Lense-Thirring effect*.]

4. [Health Warning: This question is not short.] Show that the expansion of the Ricci tensor, to second order around Minkowski spacetime, is

$$R_{\mu\nu}^{(2)}[h] = \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} \\ + \partial^\sigma h^\rho{}_\nu\partial_{[\sigma}h_{\rho]\mu} - \frac{1}{4}\partial^\rho h\partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho}$$

Hence show that, to quadratic order, and ignoring total derivatives, the expansion of the Einstein-Hilbert action gives the Fierz-Pauli action,

$$S_{FP} = \frac{1}{8\pi G} \int d^4x \left[ -\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\nu h^{\rho\mu} + \frac{1}{4}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\mu h \right]$$

Confirm that varying this action with respect to  $h_{\mu\nu}$  gives the linearised Einstein equations  $G_{\mu\nu} = 0$ .

5. A gravitational wave has polarisation  $H_{\mu\nu}$  satisfying the de Donder gauge condition  $k^\mu H_{\mu\nu} = 0$ . Show that one can use residual gauge transformations to impose the transverse-traceless condition

$$H_{0\mu} = H^\mu{}_\mu = 0$$

6\*. Two bodies, with masses  $m_1$  and  $m_2$ , move in a circular Newtonian orbit in the  $(x, y)$ -plane. Choosing coordinates so that the centre of mass sits at the origin, the bodies have positions  $\mathbf{x}_1 = (r_1 \cos \omega t, r_1 \sin \omega t, 0)$  and  $\mathbf{x}_2 = (-r_2 \cos \omega t, -r_2 \sin \omega t, 0)$  where  $m_1 r_1 = m_2 r_2 = \mu R$  and  $\mu = m_1 m_2 / (m_1 + m_2)$  is the reduced mass. The objects orbit with frequency

$$\omega^2 = \frac{G(m_1 + m_2)}{R^3}$$

i.) Treating the bodies as non-relativistic point masses, compute the corresponding energy-momentum tensor, the second moment of the energy distribution  $I_{ij}$ , and the metric perturbation  $\bar{h}_{ij}$ .

Show that the power emitted in gravitational waves is

$$\mathcal{P} = \frac{32}{5} \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{R^5}$$

ii.) The energy of the Newtonian orbit is  $E = -Gm_1m_2/2R$ . As the system emits gravitational radiation, the two bodies get closer. Set  $dE/dt = -\mathcal{P}$  to get an expression for how the radius of the orbit shrinks with time. Show that if the two bodies sit at distance  $R_0$  at time  $t = 0$ , then they collide at time

$$T = \frac{5}{256} \frac{R_0^4}{G^3 m_1 m_2 (m_1 + m_2)}$$

iii.) Consider two black holes in a circular orbit, each with mass  $30M_\odot$  where  $M_\odot$  is the mass of the Sun, separated by the distance between the Earth and Sun. How long do they take to collide? How far apart are such black holes when the merger time is a year?

[Some numbers: the Schwarzschild radius of the Sun is  $2GM_\odot \approx 3\text{km}$ . The Earth-Sun distance is around  $1.5 \times 10^8\text{km}$ . In thinking about your answers, you might wish to know that the age of the universe is about  $10^{10}$  years and the radius of the Sun is about  $7 \times 10^5\text{km}$ .]

iv.) As the radius decreases, the frequency of the orbit, and hence of the emitted gravitational waves, increases. Show that by measuring  $\omega$  and  $\dot{\omega}$ , one can extract a characteristic mass scale for the binary system, known as the *chirp mass*,

$$M_{\text{chirp}} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$