Problem Set 2

Inflation and Perturbation Theory

Warmup Questions

(a) What is the horizon problem? How does inflation solve it?

(b) What are the conditions for successful slow-roll inflation?

(c) What is the gauge problem?

(d) What are adiabatic fluctuations?

(e) Explain the relevance of the conservation of the constant density curvature perturbation ζ and comoving curvature perturbation $\mathcal{R}$ on superhorizon scales.

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1. Scalar Field Dynamics

The Lagrangian for a scalar field in a curved spacetime is

$$L = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] ,$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor.

(a) Evaluate the scalar field Lagrangian for a homogeneous field $\phi = \phi(t)$ in an FRW spacetime. From the Euler-Lagrange equation determine the equation of motion for the scalar field.

(b) Near the minimum of the inflaton potential, we can write $V(\phi) = \frac{1}{2}m^2\phi^2 + \cdots$. Making the ansatz $\phi(t) = a^{-3/2}(t)\chi(t)$, show that the equation of motion becomes

$$\ddot{\chi} + \left(m^2 - \frac{3}{2}\dot{H} - \frac{9}{4}H^2\right)\chi = 0 .$$

Assuming that $m^2 \gg H^2 \sim \dot{H}$, find $\phi(t)$. What does this result imply for the evolution of the energy density during the oscillating phase after inflation?

2. Slow-Roll Inflation

The equations of motion of the homogeneous part of the inflaton are

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \ , \qquad 3M_{\rm pl}^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V \ .$$

(a) For the potential $V(\phi) = \frac{1}{2}m^2\phi^2$, use the slow-roll approximation to obtain the inflationary solutions

$$\phi(t) = \phi_I - \sqrt{\frac{2}{3}} m M_{\rm pl}\, t \ , \qquad a(t) = a_I \exp\left[\frac{\phi_I^2 - \phi^2(t)}{4M_{\rm pl}^2}\right] \ ,$$

where $\phi_I > 0$ is the field value at the start of inflation ($t_I \equiv 0$).

(b) What is the value of ϕ when inflation ends? Find an expression for the number of e-folds. If $V(\phi_I) \sim M_{\rm pl}^4$, estimate the total number of e-folds of inflation.

3. Curvature Perturbations

In class we showed that ζ is conserved on superhorizon scales. Explicitly show the same is true for $\mathcal{R}$

4. Cosmological Gravitational Waves

(a) The line element of a FRW metric with tensor (gravitational wave) perturbations is

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + (\delta_{ij} + 2\hat{E}_{ij})\mathrm{d}x^i\mathrm{d}x^j\right] \ ,$$

where $\hat{E}_{ij}$ is symmetric, trace-free and transverse. Working to linear order in $\hat{E}_{ij}$, show that the non-zero connection coefficients are

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} \ , \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + 2\mathcal{H}h_{ij} + \hat{E}'_{ij} \ , \\
\Gamma^i_{j0} &= \mathcal{H}\delta^i_j + \hat{E}^{i\prime}_j \ , \\
\Gamma^i_{jk} &= \partial_j h^i{}_k + \partial_k \hat{E}^i{}_j - \delta^{il}\partial_l \hat{E}_{jk} \ .
\end{aligned}$$

(b)* Show that the perturbation to the Einstein tensor has non-zero components

$$\delta G_{ij} = \hat{E}''_{ij} - \nabla^2\hat{E}_{ij} + 2\mathcal{H}h'_{ij} - 2\hat{E}_{ij}(2\mathcal{H}' + \mathcal{H}^2) \ .$$

[Hint: Convince yourself that the Ricci scalar has no tensor perturbations at first order.]

(c) Show further that for tensor perturbations, the non-zero perturbations to the energy-momentum tensor are

$$\delta\hat{T}_{ij} = 2a^2\bar{P}\hat{E}_{ij} - a^2\hat{\Pi}_{ij} \ ,$$

where $\hat{\Pi}_{ij}$ is the anisotropic stress tensor.

(d) Combine these results, and the zeroth-order Friedmann equation, to show that the perturbed Einstein equation reduces to

$$\hat{E}''_{ij} + 2\mathcal{H}\hat{E}'_{ij} - \nabla^2 \hat{E}_{ij} = -8\pi G a^2 \hat{\Pi}_{ij} \ .$$

(e) For the case where $\nabla^2 \hat{E}_{ij} = -k^2 \hat{E}_{ij}$ (i.e. a Fourier mode of the metric perturbation), and assuming the anisotropic stress can be ignored, show that

$$\hat{E}_{ij} \propto \frac{k\tau \cos(k\tau) - \sin(k\tau)}{(k\tau)^3}$$

is a solution for a matter-dominated universe ($a \propto \tau^2$).

(f) Show that the solution tends to a constant for $k\tau \ll 1$ and argue that such a constant solution always exists for super-Hubble gravitational waves irrespective of the equation of state of the matter. For the specifc solution above, show that well inside the Hubble radius it oscillates at (comoving) frequency k and with an amplitude that falls as $1/a$. (This behaviour is also general and follows from a WKB solution of the Einstein equation.)