Mathematical Tripos Part III Lent 2025
Black Holes: Examples Sheet 4 J.E. Santos

1. This example works through the proof of the zeroth law of black hole mechanics. Let $\mathcal{N}$ be a Killing horizon of a Killing vector field ξ with surface gravity κ.

   (a) If we know that $A = 0$ on $\mathcal{N}$ for some tensor $A_{a_1 \ldots a_p}$ then $A \cdot B \equiv A_{a_1 \ldots a_p} B^{a_1 \cdots a_p} = 0$ on $\mathcal{N}$ for any tensor $B^{a_1 \cdots a_p}$. Hence $\mathcal{N}$ is a surface of constant $A \cdot B$, so $d(A \cdot B)$ is normal to $\mathcal{N}$ hence $\xi \wedge d(A \cdot B) = 0$ on $\mathcal{N}$. (i) Show that this implies $\xi_{[a} \nabla_{b]} A_{c_1 \ldots c_p} = 0$ on $\mathcal{N}$. (ii) Taking $A_a = \xi^b \nabla_b \xi_a - \kappa \xi_a$, use this (and the formula $\nabla_a \nabla_b \xi_c = R^d{}_{abc} \xi_d$) to show

$$\xi_a \xi_{[d} \nabla_{c]} \kappa + \kappa \xi_{[d} \nabla_{c]} \xi_a = \left(\xi_{[d} \nabla_{c]} \xi^b \right) \nabla_b \xi_a + \xi^b \xi_{[d} R^e{}_{c]ba} \xi_e \qquad \text{on } \mathcal{N}. \tag{1}$$

   (b) Using Frobenius' theorem, show that

$$\xi_c \nabla_a \xi_b = -2 \xi_{[a} \nabla_{b]} \xi_c \qquad \text{on } \mathcal{N}. \tag{2}$$

   Hence show that $\left(\xi_{[d} \nabla_{c]} \xi^b \right) \nabla_b \xi_a = \kappa \xi_{[d} \nabla_{c]} \xi_a$ on $\mathcal{N}$, so equation (1) reduces to

$$\xi_a \xi_{[d} \nabla_{c]} \kappa = \xi^b \xi_{[d} R^e{}_{c]ba} \xi_e \qquad \text{on } \mathcal{N}. \tag{3}$$

   (c) Set $A_{abc} = \xi_c \nabla_a \xi_b + 2 \xi_{[a} \nabla_{b]} \xi_c$ and use the result of (a)(i) and equation (2) to show that

$$\xi_c \xi_{[d} \nabla_{e]} \nabla_a \xi_b = -2 \left(\xi_{[d} \nabla_{e]} \nabla_{[b} \xi_{|c|} \right) \xi_{a]} \qquad \text{on } \mathcal{N}.$$

   and hence

$$\xi_c \xi_{[d} R^f{}_{e]ab} \xi_f = 2 \xi_{[d} R^f{}_{e]c[b} \xi_{a]} \xi_f \qquad \text{on } \mathcal{N}.$$

   (d) Contract this equation on the indices c and e, show that the LHS vanishes and the resulting equation can be written

$$-\xi_{[a} R_{b]}{}^f \xi_f \xi_d = \xi_{[a} R^f{}_{b]cd} \xi^c \xi_f \qquad \text{on } \mathcal{N}.$$

   Hence show that equation (3) reduces to

$$\xi_{[d} \nabla_{c]} \kappa = -\xi_{[d} R_{c]}{}^f \xi_f \qquad \text{on } \mathcal{N}.$$

   As (will be) explained in lectures, if the Einstein equation and the dominant energy condition are satisfied, then the RHS vanishes and hence κ is constant on the horizon.

2. In a stationary, axisymmetric, asymptotically flat, black hole spacetime, let Σ denote an asymptotically flat spacelike hypersurface that intersects $\mathcal{H}^+$ in a 2-sphere H. Let $\xi = k + \Omega_H m$ be the Killing field normal to the horizon. By considering the expression (for appropriate choices of orientation)

$$\int_{S^2_\infty} \star d\xi - \int_H \star d\xi = \int_\Sigma d \star d\xi \,,$$

   derive the *Smarr relation*

$$M = -\int_\Sigma \star J' + 2\Omega_H J + \frac{\kappa A}{4\pi} \,,$$

   where $J'_a \equiv -2 \left[T_{ab} - (1/2) T g_{ab} \right] \xi^b$.

3. Let (M, g, F) be a stationary, axisymmetric, asymptotically flat, black hole solution of the Einstein-Maxwell equations. Assume that it is possible to choose a gauge so that

$$\mathcal{L}_k A = \mathcal{L}_m A = 0 \ ,$$

The *co-rotating electric potential* is defined by

$$\Phi = -\xi^a A_a \ .$$

Use Einstein equation, and the fact that $R_{ab}\xi^a\xi^b = 0$ on a Killing horizon, to show that Φ is constant on the horizon. In particular, show that, for a choice of gauge for which $\Phi = 0$ at infinity, the value of Φ on the horizon is

$$\Phi_H = \frac{Qr_+}{r_+^2 + a^2}$$

for an electrically charged Kerr-Newman black hole, where $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$.

4. Let $(\mathcal{M}, g, F)$ be an asymptotically flat, stationary, axisymmetric, black hole solution of the Einstein-Maxwell equations and let Σ be a spacelike hypersurface with one boundary at spatial infinity and an internal boundary, H, at the event horizon of a black hole of charge Q. Show that the Smarr relation can be written

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q \ .$$

[Hint: $\mathcal{L}_\xi(F^{ab}A_b) = 0$]

5. Use the canonical commutation relations to derive $[a(f), a(g)^\dagger] = (f, g)$ and $[a(f), a(g)] = 0$.

6. Let k be a future-directed timelike Killing vector field in a globally hyperbolic spacetime. Show that $\mathcal{L}_k$ is anti-hermitian with respect to the Klein-Gordon inner product. Show that positive frequency solutions have positive Klein-Gordon norm.

7. Consider a 2d cosmological spacetime with metric $ds^2 = A(\eta)^2\left(-d\eta^2 + dx^2\right)$. Assume that $a(\eta)$ takes a constant positive value A_- or A_+ for $\eta < 0$ and $\eta > 0$ respectively. (The metric is discontinuous, but we can regard it as an approximation to a metric in which A varies smoothly from A_- to A_+ in a very short time. Birrell and Davies (section 3.4) discuss this smooth case.) Let M_-, M_+ denote the regions $\eta < 0$ and $\eta > 0$ respectively. Obtain the normalized positive frequency modes of the massive Klein-Gordon equation in $M_\pm$. Assume that the scalar field is in the vacuum state in M_-. What is the expected number of particles with wavenumber k in M_+?

8. A scalar field Φ in the Kruskal spacetime satisfies the Klein-Gordon equation $\nabla^2\Phi - \mu^2\Phi = 0$. Assume that, in static Schwarzschild coordinates, Φ takes the form

$$\Phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{r}\phi_{lm}(t,r)Y_{\ell m}(\theta,\phi)$$

where $Y_{\ell m}$ are spherical harmonics. (i) Show that ϕ_{lm} satisfies the equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{d^2}{dr_*^2} + V_l(r_*)\right]\phi_{lm} = 0 \qquad V_l(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right)$$

For a mode of definite frequency ω: $\phi_{lm} = e^{-i\omega t} R_{\omega lm}(r)$ this reduces to the radial equation:

$$\left[-\frac{d^2}{dr_*^2} + V_l(r_*) \right] R_{\omega lm} = \omega^2 R_{\omega lm}$$

An instability of the black hole with respect to scalar field perturbations would be indicated by the existence of a mode that is regular on $\mathcal{H}^+$ and decaying as $r \to \infty$, with $\omega = \omega_1 + i\omega_2$ and $\omega_2 > 0$ (so that the mode grows exponentially in time). Show that (i) the operator on the LHS of the radial equation is self-adjoint for such modes; (ii) no such instability exists.

9. Use the fact that a Schwarzschild black hole radiates at the Hawking temperature $T_H = 1/(8\pi M)$ (in units for which $\hbar$, G, c, and Boltzmann's constant all equal 1) to show that the thermal equilibrium of a black hole with an infinite reservoir of radiation at temperature T_H is unstable.

   A finite reservoir of radiation of volume V at temperature T has an energy, E_{res} and entropy, S_{res} given by $E_{res} = \sigma V T^4$, $S_{res} = \frac{4}{3}\sigma V T^3$ where σ is a constant. A Schwarzschild black hole of mass M is placed in the reservoir. Assuming that the black hole has entropy $S_{BH} = 4\pi M^2$ show that the total entropy $S = S_{BH} + S_{res}$ is extremized for fixed total energy $E = M + E_{res}$, when $T = T_H$, Show that the extremum is a maximum if and only if $V < V_c$, where the critical value of V is

$$V_c = \frac{2^{20}\pi^4 E^5}{5^5 \sigma}$$

   What happens as V passes from $V < V_c$ to $V > V_c$, or vice-versa?

10. The specific heat of a charged black hole of mass M, at fixed charge Q, is

$$C \equiv T_H \left. \frac{\partial S_{BH}}{\partial T_H} \right|_Q ,$$

   where T_H is its Hawking temperature and S_{BH} its entropy. Assuming that the entropy of a black hole is given by $S_{BH} = \frac{1}{4}A$, where A is the area of the event horizon, show that the specific heat of a Reissner-Nordstrom black hole is

$$C = \frac{2S_{BH}\sqrt{M^2 - Q^2}}{(M - 2\sqrt{M^2 - Q^2})} .$$

   Hence show that C^{-1} changes sign when M passes through $2|Q|/\sqrt{3}$.

   Repeat the previous question for a Reissner-Nordstrom black hole. Specifically, show that the critical reservoir volume, V_c, is infinite for $|Q| \le M \le 2|Q|/\sqrt{3}$. Why is this result to be expected from your previous result for C?