

**Mathematical Tripos Part IA**  
**Vectors and Matrices**

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**Example Sheet 1**

1. Let  $D$  be the interior of the circle  $|z - 1 - i| = 1$ . Show, by using suitable inequalities for  $|z_1 \pm z_2|$ , that if  $z \in D$  then

$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.$$

Obtain the same result geometrically [start by considering the line through the centre of the circle and the point 3].

2. Given  $|z| = 1$  and  $\arg z = \theta$ , find both algebraically and geometrically the modulus-argument forms of

$$(i) \quad 1 + z, \quad (ii) \quad 1 - z.$$

Show that the locus of  $w$  as  $z$  varies with  $|z| = 1$ , where  $w$  is given by

$$w^2 = \frac{1 - z}{1 + z},$$

is a pair of straight lines.

3. Consider a triangle in the complex plane with vertices at 0,  $z_1$  and  $z_2$ . Write down an expression for the general point on the median through  $z_1$ , and a similar expression for the general point on the median through  $z_2$ . Show that the three medians of the triangle are concurrent.

4. Express

$$I = \frac{z^5 - 1}{z - 1}$$

as a polynomial in  $z$ . By considering the complex fifth root of unity  $\omega$ , obtain the four factors of  $I$  linear in  $z$ . Hence write  $I$  as the product of two real quadratic factors. By considering the term in  $z^2$  in the identity so obtained for  $I$ , show that

$$4 \cos \frac{\pi}{5} \sin \frac{\pi}{10} = 1.$$

5. Find all complex numbers  $z$  that satisfy  $\sin z = 2$ .
6. (a) Let  $z, a, b \in \mathbb{C}$  ( $a \neq b$ ) correspond to points  $P, A, B$  in the Argand diagram. Let  $C_\lambda$  be the locus of  $P$  defined by

$$|PA|/|PB| = \lambda,$$

where  $\lambda$  is a fixed real positive constant. Show that  $C_\lambda$  is a circle if  $\lambda \neq 1$ , and find its centre and radius. What happens if  $\lambda = 1$ ?

- (b) For the case  $a = -b = p$ ,  $p \in \mathbb{R}$ , and for each fixed  $\mu \in \mathbb{R}$ , show that

$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$

is a circle passing through  $A$  and  $B$  with its centre on the perpendicular bisector of  $AB$ .

Show that the circles  $C_\lambda$  and  $S_\mu$  intersect orthogonally for all  $\lambda, \mu$ .

7. Show by vector methods that the altitudes of a triangle are concurrent.

[Hint: let the altitudes  $AD, BE$  of  $\triangle ABC$  meet at  $H$ , and show that  $CH$  is perpendicular to  $AB$ .]

8. In three dimensions, the vectors  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{a}$  satisfy

$$\mathbf{x} + \mathbf{y}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{a}.$$

Show that

$$(\mathbf{x} \cdot \mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2}.$$

Use an inequality involving  $\mathbf{x} \cdot \mathbf{y}$  and the lengths of  $\mathbf{x}$  and  $\mathbf{y}$  to deduce that

$$|\mathbf{x}|(1 + |\mathbf{y}|^2) \geq |\mathbf{a}| \geq |\mathbf{x}|.$$

Explain the circumstances in which either of the inequalities above become equalities, and describe the relation between  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{a}$  in these circumstances.

9. (a) In  $\triangle ABC$ , let  $\overrightarrow{AB} = \mathbf{u}$ ,  $\overrightarrow{BC} = \mathbf{v}$  and  $\overrightarrow{CA} = \mathbf{w}$ . Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u},$$

and hence obtain the sine rule for  $\triangle ABC$ .

- (b) Given any three vectors  $\mathbf{p}$ ,  $\mathbf{q}$ ,  $\mathbf{r}$  such that

$$\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{r} = \mathbf{r} \times \mathbf{p},$$

and  $|\mathbf{p} \times \mathbf{q}| \neq 0$ , show that

$$\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}.$$

10. Interpret geometrically the equations

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b},$$

and

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are the position vectors of points in three dimensions and  $\lambda$ ,  $\mu$  are scalar parameters that take any real values. For each equation, obtain an equivalent form that does not involve parameters  $\lambda$  or  $\mu$ . How do your equations behave when the points  $\mathbf{a}$ ,  $\mathbf{b}$  or  $\mathbf{c}$  are interchanged?

11. (a) Using the identity  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ , show that

$$\begin{aligned} \text{(i)} \quad & (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \\ \text{(ii)} \quad & \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}. \end{aligned}$$

Relate the case when  $\mathbf{c} = \mathbf{a}$  and  $\mathbf{d} = \mathbf{b}$  in (i) to a well-known trigonometric identity.

Evaluate  $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$  in two ways and compare the results to find an explicit linear combination of the four vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$  that is zero.

- (b) Given that  $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ , show that

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.$$

12. Let  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$  be fixed vectors in three dimensions. For each of the following equations, find all solutions for  $\mathbf{r}$ :

$$\text{(i)} \quad \mathbf{r} + \mathbf{r} \times \mathbf{d} = \mathbf{c}; \quad \text{(ii)} \quad \mathbf{r} + (\mathbf{r} \cdot \mathbf{a})\mathbf{b} = \mathbf{c}.$$

[In (ii), consider separately the cases  $\mathbf{a} \cdot \mathbf{b} \neq -1$  and  $\mathbf{a} \cdot \mathbf{b} = -1$ .]

13. The vectors  $\mathbf{e}_r$ ,  $\mathbf{e}_\theta$ ,  $\mathbf{e}_\phi$  are defined in terms of the standard basis vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  by

$$\begin{aligned} \mathbf{e}_r &= \cos \phi \sin \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \theta \mathbf{k}, \\ \mathbf{e}_\theta &= \cos \phi \cos \theta \mathbf{i} + \sin \phi \cos \theta \mathbf{j} - \sin \theta \mathbf{k}, \\ \mathbf{e}_\phi &= -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} \end{aligned}$$

where  $\theta$  and  $\phi$  are real numbers. Show, as efficiently as possible, that  $\mathbf{e}_r$ ,  $\mathbf{e}_\theta$ ,  $\mathbf{e}_\phi$  are an orthonormal right-handed set.

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