

Mathematical Tripos Part IA
Vectors and Matrices

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Example Sheet 1

1. Let D be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in D$ then

$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.$$

Obtain the same result geometrically [start by considering the line through the centre of the circle and the point 3].

2. Given $|z| = 1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of

$$(i) \quad 1 + z, \quad (ii) \quad 1 - z.$$

Show that the locus of w as z varies with $|z| = 1$, where w is given by

$$w^2 = \frac{1 - z}{1 + z},$$

is a pair of straight lines.

3. Consider a triangle in the complex plane with vertices at 0, z_1 and z_2 . Write down an expression for the general point on the median through z_1 , and a similar expression for the general point on the median through z_2 . Show that the three medians of the triangle are concurrent.

4. Express

$$I = \frac{z^5 - 1}{z - 1}$$

as a polynomial in z . By considering the complex fifth root of unity ω , obtain the four factors of I linear in z . Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I , show that

$$4 \cos \frac{\pi}{5} \sin \frac{\pi}{10} = 1.$$

5. Find all complex numbers z that satisfy $\sin z = 2$.
6. (a) Let $z, a, b \in \mathbb{C}$ ($a \neq b$) correspond to points P, A, B in the Argand diagram. Let C_λ be the locus of P defined by

$$|PA|/|PB| = \lambda,$$

where λ is a fixed real positive constant. Show that C_λ is a circle if $\lambda \neq 1$, and find its centre and radius. What happens if $\lambda = 1$?

- (b) For the case $a = -b = p$, $p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that

$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$

is a circle passing through A and B with its centre on the perpendicular bisector of AB .

Show that the circles C_λ and S_μ intersect orthogonally for all λ, μ .

7. Show by vector methods that the altitudes of a triangle are concurrent.

[Hint: let the altitudes AD, BE of $\triangle ABC$ meet at H , and show that CH is perpendicular to AB .]

8. In three dimensions, the vectors \mathbf{x}, \mathbf{y} and \mathbf{a} satisfy

$$\mathbf{x} + \mathbf{y}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{a}.$$

Show that

$$(\mathbf{x} \cdot \mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2}.$$

Use an inequality involving $\mathbf{x} \cdot \mathbf{y}$ and the lengths of \mathbf{x} and \mathbf{y} to deduce that

$$|\mathbf{x}|(1 + |\mathbf{y}|^2) \geq |\mathbf{a}| \geq |\mathbf{x}|.$$

Explain the circumstances in which either of the inequalities above become equalities, and describe the relation between \mathbf{x} , \mathbf{y} and \mathbf{a} in these circumstances.

9. (a) In $\triangle ABC$, let $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{CA} = \mathbf{w}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u},$$

and hence obtain the sine rule for $\triangle ABC$.

- (b) Given any three vectors \mathbf{p} , \mathbf{q} , \mathbf{r} such that

$$\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{r} = \mathbf{r} \times \mathbf{p},$$

and $|\mathbf{p} \times \mathbf{q}| \neq 0$, show that

$$\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}.$$

10. Interpret geometrically the equations

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b},$$

and

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where \mathbf{a} , \mathbf{b} and \mathbf{c} are the position vectors of points in three dimensions and λ , μ are scalar parameters that take any real values. For each equation, obtain an equivalent form that does not involve parameters λ or μ . How do your equations behave when the points \mathbf{a} , \mathbf{b} or \mathbf{c} are interchanged?

11. (a) Using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, show that

$$\begin{aligned} \text{(i)} \quad & (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \\ \text{(ii)} \quad & \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}. \end{aligned}$$

Relate the case when $\mathbf{c} = \mathbf{a}$ and $\mathbf{d} = \mathbf{b}$ in (i) to a well-known trigonometric identity.

Evaluate $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ in two ways and compare the results to find an explicit linear combination of the four vectors \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} that is zero.

- (b) Given that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, show that

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.$$

12. Let \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} be fixed vectors in three dimensions. For each of the following equations, find all solutions for \mathbf{r} :

$$\text{(i)} \quad \mathbf{r} + \mathbf{r} \times \mathbf{d} = \mathbf{c}; \quad \text{(ii)} \quad \mathbf{r} + (\mathbf{r} \cdot \mathbf{a})\mathbf{b} = \mathbf{c}.$$

[In (ii), consider separately the cases $\mathbf{a} \cdot \mathbf{b} \neq -1$ and $\mathbf{a} \cdot \mathbf{b} = -1$.]

13. The vectors \mathbf{e}_r , \mathbf{e}_θ , \mathbf{e}_ϕ are defined in terms of the standard basis vectors \mathbf{i} , \mathbf{j} , \mathbf{k} by

$$\begin{aligned} \mathbf{e}_r &= \cos \phi \sin \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \theta \mathbf{k}, \\ \mathbf{e}_\theta &= \cos \phi \cos \theta \mathbf{i} + \sin \phi \cos \theta \mathbf{j} - \sin \theta \mathbf{k}, \\ \mathbf{e}_\phi &= -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} \end{aligned}$$

where θ and ϕ are real numbers. Show, as efficiently as possible, that \mathbf{e}_r , \mathbf{e}_θ , \mathbf{e}_ϕ are an orthonormal right-handed set.

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