Mathematical Tripos Part IA **Michaelmas Term 2023**
Vectors and Matrices **Prof. N. Peake**

Example Sheet 2

1. In the following, the indices i, j, k, ℓ take the values $1, 2, 3$ and the summation convention applies.

(a) Simplify the following expressions:

$$\delta_{ij} v_j\,, \quad \delta_{ij}\delta_{jk}\,, \quad \delta_{ij}\delta_{ji}\,, \quad \delta_{ij} v_i v_j\,, \quad \varepsilon_{ijk}\delta_{jk}\,, \quad \varepsilon_{ijk} v_j v_k\,, \quad \varepsilon_{ijk}\varepsilon_{ij\ell}\,, \quad \varepsilon_{ijk}\varepsilon_{ikj}\,.$$

(b) Given that $A_{ij} = \varepsilon_{ijk}\, a_k$ (for all i, j), show that $2a_k = \varepsilon_{kij}\, A_{ij}$ (for all k).

(c) Show that $\varepsilon_{ijk}\, S_{ij} = 0$ (for all k) if and only if $S_{ij} = S_{ji}$ (for all i, j).

2. For vectors in $\mathbb{R}^3$, simplify $\varepsilon_{ijk}\,(\mathbf{a} \times \mathbf{b})_k$ and deduce a standard formula for $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$.

(a) Let $\mathbf{m}, \mathbf{u}$ and $\mathbf{a}$ be fixed vectors in $\mathbb{R}^3$ such that $\mathbf{m} \cdot \mathbf{u} = 0$ and $\mathbf{a} \cdot \mathbf{u} \neq 0$. Show that the line $\mathbf{r} \times \mathbf{u} = \mathbf{m}$ meets the plane $\mathbf{r} \cdot \mathbf{a} = \kappa$ (a constant) in the point

$$\mathbf{r} \;=\; \frac{\mathbf{a} \times \mathbf{m} \,+\, \kappa\, \mathbf{u}}{\mathbf{a} \cdot \mathbf{u}}\,.$$

Explain clearly the geometrical meaning of the condition $\mathbf{a} \cdot \mathbf{u} \neq 0$.

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$ with $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$. Show that the planes $\mathbf{r} \cdot \mathbf{a} = \kappa$ and $\mathbf{r} \cdot \mathbf{b} = \rho$ (where κ, ρ are constants) intersect in the line

$$\mathbf{r} \times (\mathbf{a} \times \mathbf{b}) \;=\; \rho\, \mathbf{a} \,-\, \kappa\, \mathbf{b}\,,$$

i.e., show that every point that lies on both planes lies on the line and, conversely, every point on the line lies on both planes. What happens if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$?

3. Show that $M_{ij} = \delta_{ij} + \varepsilon_{ijk} n_k$ and $N_{ij} = \delta_{ij} - \varepsilon_{ijk} n_k + n_i n_j$ obey $N_{ij} M_{jk} = 2\delta_{ik}$, if $n_i n_i = 1$ (indices take values $1, 2, 3$ and the summation convention applies). Verify that

$$\mathbf{y} \;=\; \mathbf{x} + \mathbf{x} \times \mathbf{n} \quad \iff \quad y_i \;=\; M_{ij}\, x_j\,,$$

where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{n}$ are vectors in $\mathbb{R}^3$ with components x_i, y_i, n_i. Use these results to find $\mathbf{x}$ in terms of $\mathbf{y}$, given that $\mathbf{n}$ is a unit vector.

4. The set X consists of six vectors in $\mathbb{R}^4$:

$$(1, 1, 0, 0),\ (1, 0, 1, 0),\ (1, 0, 0, 1),\ (0, 1, 1, 0),\ (0, 1, 0, 1),\ (0, 0, 1, 1).$$

Find two different subsets Y of X whose members are linearly independent, each of which yields a linearly dependent subset of X whenever any element $\mathbf{v} \in X$ with $\mathbf{v} \notin Y$ is adjoined to Y.

5. Let V be the set of all vectors $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ $(n \geqslant 4)$ such that their components satisfy

$$x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-3\,.$$

Show that V is a subspace of $\mathbb{R}^n$, and find a basis for V.

6. State the Cauchy-Schwarz inequality for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and give a necessary and sufficient condition for equality to hold.

(a) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, show that

$$x^2 + y^2 + z^2 \;\geqslant\; yz + zx + xy\,, \quad \text{for any real numbers } x,\, y,\, z\,.$$

(b) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that

$$3(x^2 + y^2 + z^2 + 4) \,-\, 2(yz + zx + xy) \,-\, 4(x + y + z) \;=\; 0$$

holds for unique real values of x, y, z, to be determined.

7. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. Identify the image and kernel (null space) of each of the following linear maps $\mathbb{R}^3 \to \mathbb{R}^3$:

$$\text{(a) } T : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}\,, \qquad \text{(b) } Q : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{n} \times \mathbf{x}\,.$$

Show that $T^2 = T$ and interpret the map T geometrically. Interpret the maps Q^2 and $Q^3 + Q$, and show that $Q^4 = T$.

8. Give a geometrical description of the images and kernels of each of the following linear maps on $\mathbb{R}^3$

(a) $T : (x, y, z) \mapsto (\, x + 2y + z,\ x + 2y + z,\ 2x + 4y + 2z\,)\,,$

(b) $S : (x, y, z) \mapsto (\, x + 2y + 3z,\ x - y + z,\ x + 5y + 5z\,)\,.$

9. A linear map $\mathbb{R}^4 \to \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto M\mathbf{x}$ where

$$M = \begin{pmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{pmatrix}.$$

Find the image and kernel of this map for all real values of a and b.

10. The linear map $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \cos\theta\,\mathbf{x} + (\mathbf{x} \cdot \mathbf{n})\,(1 - \cos\theta)\,\mathbf{n} - \sin\theta\,(\mathbf{x} \times \mathbf{n}) \qquad (*)$$

is a rotation by angle θ in a positive sense about the unit vector $\mathbf{n}$. Check this in the case $\mathbf{n} = (0, 0, 1)$. Show that the expression given above for a general rotation can be written $\mathbf{x}' = R\mathbf{x}$, where R is a matrix with entries R_{ij} that should be found explicitly (in terms of θ, n_i, δ_{ij}, ε_{ijk}). Hence show that

$$R_{ii} = 2\cos\theta + 1\,, \qquad \varepsilon_{ijk}R_{jk} = -2n_i \sin\theta\,.$$

Determine θ and $\mathbf{n}$ for the rotation given by the matrix

$$R = \tfrac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}.$$

11. (a) Give examples of 2×2 real matrices representing the following types of transformations in $\mathbb{R}^2$:

(i) reflection; (ii) dilation (or scaling); (iii) shear; and (iv) rotation.

Which of these types of transformation are always represented by a 2×2 matrix with determinant $+1$?

For which types (i)-(iv) do transformations A and B of the same type obey $AB = BA$ in general?

(b) A linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{x} \mapsto \mathbf{x}' = M\mathbf{x}$ is defined by $z \mapsto z' = cz$ where $z = x_1 + ix_2$, $z' = x'_1 + ix'_2$ and $c = a + ib$ is a fixed complex number. Find the real 2×2 matrix M in terms of a and b.

Which types of transformations (i)-(iv) can be obtained for particular choices of $c = a + ib$?

12. Let $R(\mathbf{n}, \theta)$ be the matrix corresponding to a rotation with angle θ and axis $\mathbf{n}$, as given in $(*)$ of question 10. Let $H(\mathbf{n})$ be the matrix corresponding to reflection in a plane through the origin with unit normal $\mathbf{n}$, as defined by

$$\mathbf{x} \mapsto \mathbf{x}' = H(\mathbf{n})\mathbf{x} = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}\,.$$

In the following, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the standard orthonormal basis vectors in $\mathbb{R}^3$.

(a) Find explicitly the matrices $R(\mathbf{i}, \frac{\pi}{2})$ and $R(\mathbf{j}, \frac{\pi}{2})$ and check that $R(\mathbf{i}, \frac{\pi}{2})R(\mathbf{j}, \frac{\pi}{2}) \neq R(\mathbf{j}, \frac{\pi}{2})R(\mathbf{i}, \frac{\pi}{2})$.

(b) Show by both algebraic and geometrical means that the map $\mathbf{x} \mapsto \mathbf{x}' = -H(\mathbf{n})\mathbf{x}$ is a rotation through an angle π about $\mathbf{n}$.

(c) Given that $\mathbf{n}_\pm = \cos\left(\tfrac{1}{2}\theta\right)\mathbf{i} \pm \sin\left(\tfrac{1}{2}\theta\right)\mathbf{j}$, prove that

$$H(\mathbf{i})\,H(\mathbf{n}_-) = H(\mathbf{n}_+)\,H(\mathbf{i}) = R(\mathbf{k}, \theta)\,,$$

and draw diagrams to explain the geometrical meaning of this result.

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