Mathematical Tripos Part IA **Michaelmas Term 2023**
Vectors and Matrices **Professor N. Peake**

Example Sheet 4

1. A square matrix A is *upper triangular* if $A_{ij} = 0$ for $i > j$. Show that the eigenvalues of such a matrix are its diagonal entries: $\lambda_i = A_{ii}$ (no sum over i).

2. Show that the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

has characteristic equation $(t-2)^3 = 0$. Explain, as simply as possible, why M is not diagonalisable.

3. Find a, b and c such that

$$\begin{pmatrix} 1/3 & 0 & a \\ 2/3 & 1/\sqrt{2} & b \\ 2/3 & -1/\sqrt{2} & c \end{pmatrix}$$

is an orthogonal matrix. Does this condition determine a, b and c uniquely?

4. Determine the eigenvalues and eigenvectors of the symmetric matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Use an identity of the form $P^{\mathrm{T}}AP = D$, where D is a diagonal matrix, to find A^{-1}.

5. Diagonalise the quadratic form in $\mathbb{R}^2$ defined by

$$\mathcal{F}(x,y) = (a\cos^2\theta + b\sin^2\theta)\,x^2 + 2(a-b)(\sin\theta\cos\theta)\,xy + (a\sin^2\theta + b\cos^2\theta)\,y^2,$$

i.e., find its eigenvalues and principal axes (a, b and θ are constants).

6. (i) A matrix A is anti-hermitian, $A^\dagger = -A$; show that the eigenvalues of A are pure-imaginary.
(ii) A matrix U is unitary, $U^\dagger U = I$; show that the eigenvalues of U have unit modulus.
(iii) In each of the cases (i) and (ii), show that eigenvectors with distinct eigenvalues are orthogonal.

7. Check, by direct calculation, that the Cayley-Hamilton Theorem holds for a general 2×2 matrix.
Find the characteristic polynomial for

$$A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$

and deduce that $A^2 = 2A - I$. Is A diagonalisable?
Show by induction that

$$A^n = \alpha_n A + \beta_n I, \quad n \geq 0,$$

for real numbers α_n and β_n. Solve the recurrence relations (difference equations) satisfied by α_n and β_n and hence find A^n explicitly.

8. Define the $m \times n$ matrix A that represents a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ with respect to general bases $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$.

(a) Taking $n = 2$, $m = 3$, let T be defined by

$$T : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad T : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}.$$

Find the matrix A with respect to the bases

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}; \qquad \mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Taking $n = m = 3$, let T be reflection in the plane $x_1 \sin\theta = x_2 \cos\theta$. Find the matrix A with respect to a convenient choice of bases (to be specified) such that $\mathbf{e}_i = \mathbf{f}_i$ $(i = 1, 2, 3)$.

(c) Taking $n = m = 2$, let T be the shear with parameter λ defined by

$$T : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \lambda \\ 1 \end{pmatrix}.$$

Find the matrix A when $\mathbf{e}_1 = \mathbf{f}_1$, and $\mathbf{e}_2 = \mathbf{f}_2$ are the standard basis vectors for $\mathbb{R}^2$; find also the matrix A' with respect to a new basis $\mathbf{e}'_1 = \mathbf{f}'_1 = -\mathbf{e}_2$ and $\mathbf{e}'_2 = \mathbf{f}'_2 = \mathbf{e}_1$. Show that $A' = R^{-1}AR$ for a rotation matrix R and comment on this result.

9. The linear map $S : \mathbb{R}^2 \to \mathbb{R}^2$ is defined in terms of its matrix A with respect to the standard basis by

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.$$

Find the matrix B for S with respect to the basis

$$\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Show that

$$B^n - I = n(B - I)$$

for all positive integers n, and hence determine A^n. Verify that $\det(A^n) = (\det A)^n$.

10. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

$$A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Hence sketch the surfaces

$$5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1.$$

11. Let Σ be the surface in $\mathbb{R}^3$ given by

$$2x^2 + 2xy + 4yz + z^2 = 1.$$

By considering a suitable real symmetric matrix, show that there is a new orthonormal basis with associated coordinates u, v, w such that Σ is given by

$$\lambda u^2 + \mu v^2 + \nu w^2 = 1,$$

for constants λ, μ, ν, to be determined. Find the minimum distance from a point on Σ to the origin. [You need not find the new basis vectors explicitly.]

12. If S is a real symmetric matrix and A is a real antisymmetric matrix, show that $A + iS$ is anti-hermitian (see question **6**, part (i), above) and deduce that

$$\det(A + iS - I) \neq 0.$$

Show that the matrix

$$U = (I + A + iS)(I - A - iS)^{-1}$$

is unitary. Find U when

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and show that it has eigenvalues $\pm(1 - i)/\sqrt{2}$.

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