

## Examples Sheet 1

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.

1. Show, from first principles, that, for non-negative integer  $n$ ,  $d^n x^n / dx^n = nx^{n-1}$ .
2. Let  $f(x) = u(x)v(x)$ . Use the definition of the derivative of a function to show that

$$\frac{df}{dx} = u \frac{dv}{dx} + \frac{du}{dx} v.$$

3. Calculate:

- (i)  $d^n(xe^x)/dx^n$  using (a) the Leibniz rule and (b) application of the product rule;
- (ii)  $d^3(\ln x)^2/dx^3$ .

4. (i) Write down or determine the Taylor series for  $f(x) = e^{ax}$  about  $x = 1$ .

- (ii) Write down or determine the Taylor series for  $\ln(1+x)$  about  $x = 0$ . Then show that

$$\lim_{k \rightarrow \infty} k \ln \left( 1 + \frac{x}{k} \right) = x$$

and deduce that

$$\lim_{k \rightarrow \infty} \left( 1 + \frac{x}{k} \right)^k = e^x.$$

5. Determine by any method the first three non-zero terms of the Taylor expansions about  $x = 0$  of:

- (i)  $(x^2 + a)^{-3/2}$ ;
- (ii)  $\ln(\cos x)$ ;
- (iii)  $\exp[-(x-a)^{-2}]$ , where  $a$  is a constant.

6. By considering the area under the curves  $y = \ln x$  and  $y = \ln(x - 1)$ , show that

$$N \ln N - N < \ln(N!) < (N + 1) \ln(N + 1) - N.$$

Hence show that

$$\frac{N!}{N + 1} < \left( \frac{N + 1}{e} \right)^N.$$

7. Show that

$$y(x) = \int_x^\infty e^{-t^2} dt$$

satisfies the differential equation  $y'' + 2xy' = 0$ .

\*8. Let  $J_n$  be the indefinite integral

$$J_n = \int \frac{x^{-n}}{(ax^2 + 2bx + c)^{\frac{1}{2}}} dx.$$

By integrating  $\int x^{-(n+1)}(ax^2 + 2bx + c)^{\frac{1}{2}} dx$  by parts, show that, for  $n \neq 0$ ,

$$ncJ_{n+1} + (2n - 1)bJ_n + (n - 1)aJ_{n-1} = -x^{-n}(ax^2 + 2bx + c)^{\frac{1}{2}}.$$

Hence evaluate

$$\int_1^2 \frac{1}{x^{5/2}(x + 2)^{\frac{1}{2}}} dx.$$

\*9. In a large population, the proportion with income between  $x$  and  $x + dx$  is  $f(x)dx$ . Express the mean (average) income  $\mu$  as an integral, assuming that any positive income is possible.

Let  $p = F(x)$  be the proportion of the population with income less than  $x$ , and  $G(x)$  be the mean (average) income earned by people with income less than  $x$ . Further, let  $\theta(p)$  be the proportion of the total income that is earned by people with income less than  $x$  as a function of the proportion  $p$  of the population that has income less than  $x$ . Express  $F(x)$  and  $G(x)$  as integrals and thence derive an expression for  $\theta(p)$ , showing that

$$\theta(0) = 0, \quad \theta(1) = 1$$

and

$$\theta'(p) = \frac{F^{-1}(p)}{\mu}, \quad \theta''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0.$$

Sketch the graph of a function  $\theta(p)$  with these properties. Deduce that, if there is any variation in income, the bottom (when ordered in terms of income) proportion  $p$  of the population receive less than  $p$  of the total income, for all positive values of  $p$ . Just how much less is quantified by the (in)famous ‘‘Gini index’’ beloved of economists, which is twice the area between the curve  $\theta(p)$  and the diagonal line connecting  $(0, 0)$  and  $(1, 1)$ .

A particular population's income is described by the exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0.$$

For some constant  $\lambda > 0$ , using your expression for  $\theta(p)$ , compute the Gini index in this case.

[Food for thought: To what extent does the Gini index capture the degree of income inequality for this distribution?]

10. For  $f(x, y) = \exp(-xy)$ , find  $(\partial f/\partial x)|_y$  and  $(\partial f/\partial y)|_x$ . Check that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Now express  $x$  and  $y$  in polar coordinates,  $x = r \cos \theta$  and  $y = r \sin \theta$ . Find  $(\partial f/\partial r)|_\theta$  and  $(\partial f/\partial \theta)|_r$  using:

- (a) the chain rule; and
- (b) by first expressing  $f$  in terms of  $r$  and  $\theta$ .

Check that the two methods give the same results.

11. If  $xyz + x^3 + y^4 + z^5 = 0$  (an implicit equation for any of the variables  $x, y, z$  in terms of the other two), find

$$\left. \frac{\partial x}{\partial y} \right|_z, \quad \left. \frac{\partial y}{\partial z} \right|_x \quad \text{and} \quad \left. \frac{\partial z}{\partial x} \right|_y$$

and show that their product is  $-1$ .

Does this result hold for an arbitrary relation  $f(x, y, z) = 0$ ?

What about  $f(x_1, x_2, \dots, x_n) = 0$ ?

12. In thermodynamics, the pressure of a system,  $p$ , can be considered as a function of the variables  $V$  (volume) and  $T$  (temperature) or as a function of  $V$  and  $S$  (entropy).

- (i) By expressing  $p(V, S)$  in the form  $p(V, S(V, T))$  evaluate

$$\left. \frac{\partial p}{\partial V} \right|_T - \left. \frac{\partial p}{\partial V} \right|_S \quad \text{in terms of} \quad \left. \frac{\partial S}{\partial V} \right|_T \quad \text{and} \quad \left. \frac{\partial S}{\partial p} \right|_V.$$

- (ii) Hence, using  $TdS = dU + pdV$  (conservation of energy with  $U$  the internal energy), show that

$$\left. \frac{\partial \ln p}{\partial \ln V} \right|_T - \left. \frac{\partial \ln p}{\partial \ln V} \right|_S = \left. \frac{\partial(pV)}{\partial T} \right|_V \left[ \frac{p^{-1} (\partial U/\partial V)|_T + 1}{(\partial U/\partial T)|_V} \right].$$

[Hint: note that  $(\partial \ln p/\partial \ln V)|_T = (V/p) (\partial p/\partial V)|_T$ .]

13. By differentiating  $I$  with respect to  $\lambda$ , show that

$$I(\lambda, \alpha) = \int_0^\infty \frac{\sin \lambda x}{x} e^{-\alpha x} dx = \tan^{-1} \left( \frac{\lambda}{\alpha} \right) + c(\alpha).$$

Show that  $c(\alpha)$  is constant (independent of  $\alpha$ ) and hence, by considering the limits  $\alpha \rightarrow \infty$  and  $\alpha \rightarrow 0$ , show that, if  $\lambda > 0$ ,

$$\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}.$$

What is the value of the integral when  $\lambda < 0$ ?

14. Let

$$f(x) = \left( \int_0^x e^{-t^2} dt \right)^2 \quad \text{and} \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{(1+t^2)} dt.$$

Show that

$$f'(x) + g'(x) = 0.$$

Deduce that

$$f(x) + g(x) = \pi/4,$$

and hence that

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

*Comments and corrections may be sent by email to [a.d.challinor@ast.cam.ac.uk](mailto:a.d.challinor@ast.cam.ac.uk)*