Mathematical Tripos Part IA

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Differential Equations A3

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DIFFERENTIAL EQUATIONS

Examples Sheet 1

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.

1. Show, from first principles, that, for non-negative integer n, $\dfrac{d}{dx}x^n = nx^{n-1}$.

2. Let $f(x) = u(x)v(x)$. Use the definition of the derivative of a function to show that

$$\frac{df}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v.$$

3. Calculate
   (i) $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(xe^x)$ using (a) the Leibniz rule and (b) application of the product rule,
   (ii) $\dfrac{\mathrm{d}^3}{\mathrm{d}x^3}\left([\ln(x)]^2\right)$.

4. (i) Write down or determine the Taylor series for $f(x) = e^{ax}$ about $x = 1$.
   (ii) Write down or determine the Taylor series for $\ln(1 + x)$ about $x = 0$. Then show that

$$\lim_{k\to\infty} k\ln(1 + x/k) = x$$

and deduce that

$$\lim_{k\to\infty} (1 + x/k)^k = e^x.$$

5. Determine by any method the first three non-zero terms of the Taylor expansions about $x = 0$ of

(i) $(x^2 + a)^{-3/2}$,

(ii) $\ln(\cos x)$,

(iii) $\exp\left\{-\dfrac{1}{(x-a)^2}\right\}$,

where a is a constant.

6. By considering the area under the curves $y = \ln x$ and $y = \ln(x-1)$, show that

$$N \ln N - N < \ln(N!) < (N+1)\ln(N+1) - N.$$

Hence show that

$$\frac{N!}{N+1} < \left(\frac{N+1}{e}\right)^N .$$

7. Show that $y(x) = \int_x^\infty e^{-t^2}\,dt$ satisfies the differential equation $y'' + 2xy' = 0$.

*8. Let J_n be the indefinite integral

$$J_n = \int \frac{x^{-n}\,\mathrm{d}x}{(ax^2 + 2bx + c)^{\frac{1}{2}}} .$$

By integrating $\int x^{-n-1}(ax^2 + 2bx + c)^{\frac{1}{2}}\,\mathrm{d}x$ by parts, show that for $n \neq 0$,

$$ncJ_{n+1} + (2n-1)bJ_n + (n-1)aJ_{n-1} = -x^{-n}(ax^2 + 2bx + c)^{\frac{1}{2}}.$$

Hence evaluate

$$\int_1^2 \frac{\mathrm{d}x}{x^{5/2}(x+2)^{\frac{1}{2}}}.$$

*9. In a large population, the proportion with income between x and $x + \mathrm{d}x$ is $f(x)\mathrm{d}x$. Express the mean (average) income μ as an integral, assuming that any positive income is possible.

Let $p = F(x)$ be the proportion of the population with income less than x, and $G(x)$ be the mean (average) income earned by people with income less than x. Further, let $\theta(p)$ be the proportion of the total income which is earned by people with income less than x as a function of the proportion p of the population which has income less than x. Express $F(x)$ and $G(x)$ as integrals and thence derive an expression for $\theta(p)$, showing that

$$\theta(0) = 0, \quad \theta(1) = 1$$

and

$$\theta'(p) = \frac{F^{-1}(p)}{\mu}, \quad \theta''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0.$$

Sketch the graph of a function $\theta(p)$ with these properties. Deduce that, if there is any variation in income, the bottom, (when ordered in terms of income) proportion p of the population receive less than p of the total income, for all positive values of p. Just how much less is quantified by the (in)famous "Gini index" beloved of economists, which is twice the area between the curve $\theta(p)$ and the diagonal line connecting $(0,0)$ and $(1,1)$.

A particular population's income is described by the exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x > 0.$$

For some constant $\lambda > 0$, using your expression for $\theta(p)$, compute the Gini index in this case. *Food for thought: To what extent does the Gini index capture the degree of income inequality for this distribution?*

10. For $f(x,y) = \exp(-xy)$, find $(\partial f/\partial x)|_y$ and $(\partial f/\partial y)|_x$. Check that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$. Find $(\partial f/\partial r)|_\theta$ and $(\partial f/\partial \theta)|_r$,

    (i) using the chain rule,

    (ii) by first expressing f in terms of the polar coordinates r, θ,

    and check that the two methods give the same results.

    [Recall: $x = r\cos\theta, \quad y = r\sin\theta.$]

11. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables x, y, z in terms of the other two), find

$$\left(\frac{\partial x}{\partial y}\right)\bigg|_z, \quad \left(\frac{\partial y}{\partial z}\right)\bigg|_x, \quad \left(\frac{\partial z}{\partial x}\right)\bigg|_y$$

    and show that their product is -1.

    Does this result hold for an arbitrary relation $f(x,y,z) = 0$?

    What about $f(x_1, x_2, \cdots, x_n) = 0$?

12. In thermodynamics, the pressure of a system, p, can be considered as a function of the variables V (volume) and T (temperature) or as a function of the variables V and S (entropy).

(i) By expressing $p(V,S)$ in the form $p(V,S(V,T))$ evaluate

$$\left(\frac{\partial p}{\partial V}\right)\Big|_T - \left(\frac{\partial p}{\partial V}\right)\Big|_S \text{ in terms of } \left(\frac{\partial S}{\partial V}\right)\Big|_T \text{ and } \left(\frac{\partial S}{\partial p}\right)\Big|_V .$$

(ii) Hence, using $TdS = dU + pdV$ (conservation of energy with U the internal energy), show that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)\Big|_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)\Big|_S = \left(\frac{\partial (pV)}{\partial T}\right)\Big|_V \left[\frac{p^{-1}\left(\partial U/\partial V\right)|_T + 1}{\left(\partial U/\partial T\right)|_V}\right] .$$

$\left[\text{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)\big|_T = \frac{V}{p}\left(\frac{\partial p}{\partial V}\right)\big|_T\right]$

13. By differentiating I with respect to λ, show that

$$I(\lambda,\alpha) = \int_0^\infty \frac{\sin \lambda x}{x} e^{-\alpha x} dx = \tan^{-1}\frac{\lambda}{\alpha} + c(\alpha).$$

Show that $c(\alpha)$ is constant (independent of α) and hence, by considering the limits $\alpha \to \infty$ and $\alpha \to 0$, show that, if $\lambda > 0$,

$$\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}.$$

What is the value of the integral when $\lambda < 0$?

14. Let $f(x) = \left[\int_0^x e^{-t^2}\mathrm{d}t\right]^2$ and let $g(x) = \int_0^1 [e^{-x^2(t^2+1)}/(1+t^2)]\mathrm{d}t$.

Show that

$$f'(x) + g'(x) = 0.$$

Deduce that

$$f(x) + g(x) = \pi/4,$$

and hence that

$$\int_0^\infty e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

Comments and corrections may be sent by email to J.R.Taylor@damtp.cam.ac.uk