Mathematical Tripos Part IA

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Differential Equations A3

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Examples Sheet 3

The starred questions are intended as extras: do them if you have time, but not at the expense of questions on later sheets

1. Find the general solutions of
(i) $y'' + 5y' + 6y = e^{3x}$,
(ii) $y'' + 9y = \cos 3x$,
(iii) $y'' - 2y' + y = (x-1)e^x$.

2. The function $y(x)$ satisfies the linear equation

$$y'' + p(x)y' + q(x)y = 0.$$

The Wronskian $W(x)$ of two independent solutions, denoted $y_1(x)$ and $y_2(x)$, is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Let $y_1(x)$ be given. Use the Wronskian to determine a first-order inhomogeneous differential equation for $y_2(x)$. Hence, show that

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1(t)^2}\, dt. \tag{$*$}$$

Show that $W(x)$ satisfies

$$\frac{dW}{dx} + p(x)W = 0.$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy'' - (1-x^2)y' - (1+x)y = 0. \tag{$\dagger$}$$

Hence, using (*) with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three non-zero terms in the power series expansion for a solution, y_2, of (†) that is independent of y_1 and satisfies $y_2(0) = 0$, $y_2''(0) = 1$.

3. Find the general solutions of
(i) $y_{n+2} + y_{n+1} - 6y_n = n^2$,
(ii) $y_{n+2} - 3y_{n+1} + 2y_n = n$,
(iii) $y_{n+2} - 4y_{n+1} + 4y_n = a^n$, where $a \neq 2$. By expressing a^n as a Taylor series about $a = 2$, find the general solution in the case $a = 2$.

4. (i) Find the solution of $y'' - y' - 2y = 0$ that satisfies $y(0) = 1$ and is bounded as $x \to \infty$.
(ii) Solve the related difference equation

$$(y_{n+1} - 2y_n + y_{n-1}) - \tfrac{1}{2}h(y_{n+1} - y_{n-1}) - 2h^2 y_n = 0 \ ,$$

and show that if $0 < h \ll 1$ the solution that satisfies $y_0 = 1$ and for which y_n is bounded as $n \to \infty$ is approximately $y_n = (1 - h + \frac{1}{2}h^2)^n$. Explain the relation with the solution of (i).

5. Show that

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) \equiv \frac{1}{r}\frac{d^2}{dr^2}(rT)$$

and hence solve the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = k^2 T, \ \text{ for } r \neq 0$$

subject to the conditions that $\lim_{r\to 0} T(r)$ is finite and $T(1) = 1$.

6. Given the solution $y_1(x)$, find a second solution of the following equations:
(i) $x(x+1)y'' + (x-1)y' - y = 0, \quad y_1(x) = (x+1)^{-1}$;
(ii) $xy'' - y' - 4x^3 y = 0, \quad y_1(x) = e^{x^2}$.

*7. The n functions $y_j(x)$ $(1 \leq j \leq n)$ are independent solutions of the equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = 0.$$

Let $\mathbf{W}$ be the $n \times n$ matrix whose i, j element W_{ij} is $y_j^{(i-1)}(x)$ (so that $\det\mathbf{W} = \mathcal{W}$, the Wronskian). Find a matrix $\mathbf{A}$, which does not explicitly involve the y_j such that

$$\mathbf{W}' = \mathbf{A}\,\mathbf{W}$$

where $\mathbf{W}'$ is the matrix whose elements are given by $(\mathbf{W}')_{ij} = W'_{ij}$. Using the identity

$$(\ln\det\mathbf{W})' = \operatorname{trace}\left(\mathbf{W}'\mathbf{W}^{-1}\right),$$

express $\mathcal{W}$ in terms of $p_1(x)$. [You can prove this identity by writing $\mathbf{W} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ where $\mathbf{D}$ is in Jordan normal form (which is upper triangular) and using trace $\mathbf{ABC} =$ trace $\mathbf{BCA}$.]

8. Let $y(x)$ satisfy the inhomogeneous equation

$$y'' - 2x^{-1}y' + 2x^{-2}y = f(x) \ . \qquad (*)$$

Set

$$\begin{pmatrix} y \\ y' \end{pmatrix} = u(x)\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} + v(x)\begin{pmatrix} y_2 \\ y_2' \end{pmatrix},$$

where $y_1(x)$ and $y_2(x)$ are two independent solutions of $(*)$ when $f(x) = 0$, and $u(x)$ and $v(x)$ are unknown functions. Obtain first-order differential equations for $u(x)$ and $v(x)$, and hence find the most general solution of $(*)$ in the case $f(x) = x \sin x$. Are the functions $u(x)$ and $v(x)$ completely determined by this procedure?

9. A large oil tanker of mass W floats on the sea of density ρ. Suppose the tanker is given a small downward displacement z. The upward force is equal to the weight of water displaced (Archimedes' Principle). If the cross-sectional area A of the tanker at the water surface is constant, show that this upward force is $g\rho Az$, and hence that

$$\ddot{z} + \frac{g\rho A}{W} z = 0 \, .$$

Suppose now that a mouse exercises on the deck of the tanker producing a vertical force $m \sin \omega t$, where $\omega = (g\rho A/W)^{1/2}$. Show that the tanker will eventually sink. In practice, as the vertical motion of the tanker increases, waves will be generated. Suppose they produce an additional damping $2k\dot{z}$. Discuss the motion for a range of values of k.

10. Find and sketch the solution of

$$\ddot{y} + y = H(t - \pi) - H(t - 2\pi),$$

where H is the Heaviside step function, subject to

$$y(0) = \dot{y}(0) = 0 \, ,$$

and with $y(t)$ and $\dot{y}(t)$ continuous at $t = \pi, 2\pi$.

11. Solve

$$y'' - 4y = \delta(x - a),$$

where δ is the Dirac delta function, subject to the condition that $y(x)$ is continuous at $x = a$ and boundary conditions that y is bounded as $|x| \to \infty$. Sketch the solution.

12. Solve

$$\ddot{y} + 2\dot{y} + 5y = 2\delta(t),$$

where δ is the Dirac delta function, given that $y = 0$ for $t < 0$. Give an example of a physical system that this describes.

*13. Show that, for suitably chosen $P(x)$, the transformation $y(x) = P(x)v(x)$ reduces the equation

$$y'' + p(x)y' + q(x)y = 0$$

to the form

$$v'' + J(x)v = 0 \, . \qquad (\dagger)$$

The sequence of functions $v_n(x)$ is defined, for a given function $J(t)$ and in a given range $0 \le x \le R$, by $v_0(x) = a + bx$ and

$$v_n(x) = \int_0^x (t - x)J(t)v_{n-1}(t)\mathrm{d}t. \quad (n \ge 1) \, .$$

Show that $v_n''(x) + J(x)v_{n-1} = 0$ $(n \ge 1)$ and deduce that $v(x) = \sum_0^\infty v_n(x)$ satisfies $(\dagger)$ with the initial conditions $v(0) = a, \; v'(0) = b$.
[N.B. You may assume that the sum which defines $v(x)$ converges sufficiently nicely to allow term-by-term differentiation. In fact, you can show by induction that if $|J(x)| < m$ and $|v_0(x)| < A$ for the range of x under consideration, then $|v_n(x)| \le Am^n x^{2n}/(2n)!$ – try it! Convergence is therefore exponentially fast.]

What does this tell us about the existence problem for general second-order linear equations with given initial conditions?

*14 *The expanding universe.* Einstein's equations for a flat isotropic and homogeneous universe can be written as :

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}, \qquad H \equiv \left(\frac{\dot{a}}{a}\right) = \left(\frac{8\pi}{3}\rho + \frac{\Lambda}{3}\right)^{1/2},$$

where a is the scale factor measuring the expansion of the universe ($\dot{a} > 0$), ρ and p are the time-dependent energy density and pressure of matter, Λ is the cosmological constant and $H > 0$ the Hubble parameter. Use these equations to establish the following: If $\Lambda \sim 0$ and $\rho + 3p > 0$ the acceleration $\ddot{a} < 0$ and the graph of $a(t)$ must be concave downward implying that at a finite time a must reach $a = 0$ (the big bang). Using the tangent of the graph at present time $t = t_0$ show that the age of the universe is bounded by $t_0 < H^{-1}(t_0)$.
Consider the physical situations of a matter dominated universe ($\Lambda, p \sim 0$) and a radiation dominated universe ($\Lambda \sim 0, p = \rho/3$). In each case, reduce the two equations above to one single differential equation for a which is homogeneous in t (invariant under $t \to \lambda t$) and then show that there is a solution of the type $a = t^\alpha$. Determine the value of α for each case and verify that $\ddot{a} < 0$.
Now consider a Λ dominated universe ($\rho, p << \Lambda$), solve the differential equation for $a(t)$ and show that it corresponds to an accelerated universe ($\ddot{a} > 0$)for $\Lambda > 0$. This could describe the universe today and/or a very early period of exponential expansion known as inflation.

Comments and corrections may be sent by email to J.R.Taylor@damtp.cam.ac.uk