**Mathematical Tripos Part IA**
**Vector Calculus, Example Sheet 1**

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*Comments and corrections to* `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

**1.** Sketch the curve in the plane given parametrically by

$$\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t), \quad 0 \le t \le 2\pi.$$

Calculate $\mathbf{x}'(t)$ at each point and hence find its total length.

**2.** A circular helix is given by

$$\mathbf{x}(t) = (a\cos t, a\sin t, ct), \quad t \in \mathbf{R}.$$

Calculate the tangent $\mathbf{t}$, curvature $\kappa$, principal normal $\mathbf{n}$, binormal $\mathbf{b}$ and torsion $\tau$. Give a sketch of the curve indicating the directions of the vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

**3.** Show that a planar curve $\mathbf{x}(t) = (x(t), y(t), 0)$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Find the minimum and maximum values of the curvature on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

**4.** If $\mathbf{a}$ is constant vector and $r = |\mathbf{x}|$, verify that

$$\nabla(r^n) = nr^{n-2}\mathbf{x}, \qquad \nabla(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}$$

using (i) first principles, (ii) Cartesian coordinates and suffix notation, (iii) cylindrical polar coordinates, (iv) spherical polar coordinates. For parts (iii) and (iv) you will need to be careful with the components of $\mathbf{a}$ with respect to each of the relevant bases.

**5.** Evaluate explicitly each of the line integrals

$$\int (x\,dx + y\,dy + z\,dz), \quad \int (y\,dx + x\,dy + dz), \quad \int \left(y\,dx - x\,dy + e^{x+y}\,dz\right),$$

along (i) the straight line path from the origin to $(1,1,1)$, and (ii) the parabolic path given parametrically by $(x,y,z) = (t,t,t^2)$ from $t = 0$ to $t = 1$.

For which of these integrals do the two paths give the same results, and why?

**6.** Consider the vector fields $\mathbf{F}(\mathbf{x}) = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G}(\mathbf{x}) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the line integrals $\int \mathbf{F}\cdot d\mathbf{x}$ and $\int \mathbf{G}\cdot d\mathbf{x}$ along the following paths, each of which consist of straight line segments joining the specified points:

  (i) $(0,0,0) \to (1,1,1)$,

  (ii) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$,

  (iii) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.

Are either of the differentials $\mathbf{F}\cdot d\mathbf{x}$ or $\mathbf{G}\cdot d\mathbf{x}$ exact?

**7.** A curve $C$ is given parametrically by

$$\mathbf{x}(t) = (\cos(\sin nt)\cos t, \cos(\sin nt)\sin t, \sin(\sin nt)), \quad 0 \le t \le 2\pi,$$

where $n$ is some fixed integer. Using spherical polar coordinates, or otherwise, sketch the curve. Show that

$$\oint_C \mathbf{F}\cdot d\mathbf{x} = 2\pi, \quad \text{where} \quad \mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

and $C$ is traversed in the direction of increasing $t$. Can $\mathbf{F}(\mathbf{x})$ be written as the gradient of a scalar function? Comment on your results.

**8.** Obtain the equation of the plane which is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

**9.** Use the substitution $x = r \cos \phi$, $y = \frac{1}{2} r \sin \phi$, to evaluate

$$\int_D \frac{x^2}{x^2 + 4y^2} \, \mathrm{d}A$$

where $D$ is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

**10.** A closed curve $C$ in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area inclosed by $C$ is $D$. Show, by calculating the integrals explicitly, that

$$\oint_C \left( x^2 y \, \mathrm{d}x + xy^2 \, \mathrm{d}y \right) = \int_D (y^2 - x^2) \, \mathrm{d}A = \frac{104}{105} a^4.$$

**11.** The region $D$ is bounded by the segments $x = 0, 0 \leq y \leq 1; y = 0, 0 \leq x \leq 1; y = 1, 0 \leq x \leq \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1 - x)$. Consider a mapping into the $(x, y)$-plane from the $(u, v)$-plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch $D$ and also the two regions in the $(u, v)$-plane which are mapped into it. Hence evaluate

$$\int_D \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}.$$

## Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

**12.** Let $F : \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Using Taylor's theorem for a suitably chosen function of one variable $f = f(t)$, show that

$$F(\mathbf{x} + \mathbf{h}) = \sum_{m=0}^{N} \frac{1}{m!} D_{\mathbf{h}}^m F(\mathbf{x}) + R_N(\mathbf{x}, \mathbf{h})$$

where $D_{\mathbf{h}} \equiv \mathbf{h} \cdot \nabla$ and $R_N$ is a remainder term you should determine. Show that $|R_N(\mathbf{x}, \mathbf{h})| = o\left(|\mathbf{h}|^N\right)$.

**13.** (*Alternative argument for change of variables formula*) Consider the change of variables $(u, v) \mapsto (x, y)$, where $x = \varphi(u, v)$ and $y = \psi(u, v)$. Show that on the curve $y = \mathrm{const}$ we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{\partial \varphi}{\partial u} + \frac{\partial \varphi}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}u}, \quad 0 = \frac{\partial \psi}{\partial u} + \frac{\partial \psi}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}u}.$$

By performing the $x$-integral first and employing a suitable change of variables, show that

$$\iint f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint F(u, v) \frac{\partial(x, y)}{\partial(u, v)} \, \mathrm{d}u \, \mathrm{d}v,$$

where $F(u, v) = f(\varphi(u, v), \psi(u, v))$. You needn't worry about the ranges of integration.

**14.** Let $f = f(x)$ be a smooth function such that $\int_1^\infty \frac{1}{x} f(x) \, \mathrm{d}x$ exists. By considering an appropriate double integral involving $f'$ and changing the order of integration, show that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x = f(0) \log \frac{b}{a}, \quad a, b > 0.$$