Vector Calculus: Example Sheet 1

Part IA, Lent Term 2026

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Vector Differential Operators

1. Obtain the equation of the plane that is tangent to the surface $z = 3x^2y\sin(\frac{1}{2}\pi x)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

2. (i) Let $\phi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\boldsymbol{\nabla}\,.\,(\phi\mathbf{v}) = \boldsymbol{\nabla}\phi\,.\,\mathbf{v} + \phi\boldsymbol{\nabla}\,.\,\mathbf{v}, \qquad \boldsymbol{\nabla}\times(\phi\mathbf{v}) = \boldsymbol{\nabla}\phi\times\mathbf{v} + \phi\boldsymbol{\nabla}\times\mathbf{v}.$$

(ii) Evaluate the divergence and curl of each of the following:

$$r\mathbf{x}, \quad (\mathbf{a}\,.\,\mathbf{x})\mathbf{b}, \quad \mathbf{a}\times\mathbf{x}, \quad \frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3},$$

where $r = |\mathbf{x}|$ and $\mathbf{a}$, $\mathbf{b}$ are constant vectors.

(iii) The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals of a family of surfaces $f(\mathbf{x}) =$ constant. Show that $\mathbf{B}\,.\,(\boldsymbol{\nabla}\times\mathbf{B}) = 0$.

(iv) The vector fields $\mathbf{F}$ and $\mathbf{G}$ are everywhere parallel, non-zero and solenoidal. Show that $\mathbf{F}\,.\,\boldsymbol{\nabla}(F/G) = 0$, where $F = |\mathbf{F}|$ and $G = |\mathbf{G}|$.

3. Use suffix notation to show that for vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$,

$$\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v}) = (\boldsymbol{\nabla}\,.\,\mathbf{v})\mathbf{u} - (\boldsymbol{\nabla}\,.\,\mathbf{u})\mathbf{v} + (\mathbf{v}\,.\,\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\,.\,\boldsymbol{\nabla})\mathbf{v}$$

and

$$(\mathbf{u}\,.\,\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\tfrac{1}{2}|\mathbf{u}|^2) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}).$$

4. Verify directly that the vector field

$$\mathbf{u}(\mathbf{x}) = \big(e^x(x\cos y + \cos y - y\sin y), e^x(-x\sin y - \sin y - y\cos y), 0\big)$$

is irrotational and express it as the gradient of a scalar field ϕ. Check also that $\mathbf{u}$ is solenoidal and show that it can be written as the curl of a vector field $\mathbf{v} = (0, 0, \psi)$ for some function ψ.

* **5.** Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a homogeneous function of degree n, i.e., $f(k\mathbf{x}) = k^n f(\mathbf{x})$ for all $k \in \mathbb{R}$. By differentiating with respect to k, or otherwise, show that $\mathbf{x}\,.\,\boldsymbol{\nabla}f = nf$.

* **6.** Suppose that $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a solenoidal vector field. Show that $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x})\times(t\mathbf{x})\,\mathrm{d}t.$$

This is an example of a *homotopy formula*, in this case for finding a vector potential for a given solenoidal field. What goes wrong if the domain of definition of $\mathbf{F}$ in $\mathbb{R}^3$ contains a hole of some kind?

Orthogonal Curvilinear Coordinates

7. The vector field $\mathbf{A}(\mathbf{x})$ is, in Cartesian, cylindrical and spherical polar coordinates respectively,

$$\mathbf{A}(\mathbf{x}) = -\tfrac{1}{2}y\mathbf{e}_x + \tfrac{1}{2}x\mathbf{e}_y = \tfrac{1}{2}\rho\mathbf{e}_\phi = \tfrac{1}{2}r\sin\theta\,\mathbf{e}_\phi$$

(where $\mathbf{e}_\phi$ has two different meanings). Compute $\boldsymbol{\nabla}\times\mathbf{A}$ in each coordinate system and check that your answers agree.

8. If $\mathbf{a}$ is a constant vector and $r = |\mathbf{x}|$, verify that $\boldsymbol{\nabla} r^n = nr^{n-2}\mathbf{x}$ using (i) Cartesian coordinates and suffix notation, (ii) Taylor's theorem in the form $\mathrm{d}f = \boldsymbol{\nabla} f \,.\, \mathrm{d}\mathbf{x}$, (iii) cylindrical polar coordinates, (iv) spherical polar coordinates. Verify also that $\boldsymbol{\nabla}(\mathbf{a}\,.\,\mathbf{x}) = \mathbf{a}$ using methods (i) and (ii).

∗ Furthermore, verify that $\boldsymbol{\nabla}(\mathbf{a}\,.\,\mathbf{x}) = \mathbf{a}$ using methods (iii) and (iv). [*Hint: you will need to be careful about the components of* $\mathbf{a}$ *with respect to the relevant bases, which are* not *constant.*]

9. (i) Use the Chain Rule to express $\partial f/\partial x$ in terms of $\partial f/\partial\rho$ and $\partial f/\partial\phi$ for a function $f(\rho,\phi,z)$ with respect to cylindrical polar coordinates. Combine this with similar calculations and with expressions for the basis vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ to show that

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z.$$

(ii) We deduce from this result that

$$\boldsymbol{\nabla} = \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z}$$

and we also know that $\partial\mathbf{e}_\rho/\partial\phi = \mathbf{e}_\phi$, $\partial\mathbf{e}_\phi/\partial\phi = -\mathbf{e}_\rho$, while all other derivatives of the basis vectors are zero. Derive an expression for $\boldsymbol{\nabla}\,.\,\mathbf{A}$ where $\mathbf{A}(\mathbf{x})$ is an arbitrary vector field given in cylindrical polars by $\mathbf{A} = A_\rho\mathbf{e}_\rho + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_z$.

∗ Also derive expressions for $\boldsymbol{\nabla}\times\mathbf{A}$ and for the Laplacian $\boldsymbol{\nabla}^2 f$ of a scalar function $f(\rho,\phi,z)$.

Differential Geometry of Curves

10. Sketch the *astroid* curve in the plane given parametrically by $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$, $0 \leqslant t \leqslant 2\pi$. Calculate $\dot{\mathbf{x}}(t)$ at each point and hence find the curve's total length.

11. A circular helix is given by

$$\mathbf{x}(t) = (a\cos t, a\sin t, ct), \quad t\in\mathbb{R}$$

where a, c are constants. Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$ and binormal $\mathbf{b}$, and show that the torsion is $\tau = c/(a^2+c^2)$. Give a sketch of the curve indicating the directions of the vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$.

12. Show that a planar curve $\mathbf{x}(t) = \big(x(t), y(t), 0\big)$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}.$$

Use this result to find the minimum and maximum values of the curvature on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and comment on the case $a = b$.

∗ **13.** The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $\kappa = |\mathbf{H}\times(\mathbf{H}\,.\,\boldsymbol{\nabla})\mathbf{H}|/|\mathbf{H}|^3$.