Vector Calculus: Example Sheet 3

Part IA, Lent Term 2026

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Integral Theorems

1. Verify Stokes' Theorem for the hemispherical shell $S = \{x^2 + y^2 + z^2 = 1, z \geqslant 0\}$ and the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, z)$.

2. Verify Stokes' Theorem for the open surface defined in cylindrical polar coordinates by $\rho + z = a$, $z \geqslant 0$, where a is a positive constant, and the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, xyz)$. [*You should find that both the surface and line integrals give* $\pm 2\pi a^2$.]

3. Calculate $\boldsymbol{\nabla} \times \mathbf{B}$ for the vector field given in cylindrical polars by $\mathbf{B}(\mathbf{x}) = \rho^{-1}\mathbf{e}_\phi$. Also find $\oint_C \mathbf{B} \,.\, \mathrm{d}\mathbf{x}$ where C is the circle $\rho = 1$, $z = 0$. Does Stokes' Theorem apply?

4. Let ϕ and ψ be scalar functions defined in a volume V and on its surface. Using an integral theorem, establish *Green's second identity*

$$\int_V (\psi \boldsymbol{\nabla}^2 \phi - \phi \boldsymbol{\nabla}^2 \psi)\, \mathrm{d}V = \int_{\partial V} \left(\psi \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial \psi}{\partial \mathbf{n}} \right) \mathrm{d}S.$$

5. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let S be the *open* surface $1 - z = x^2 + y^2$, $0 \leqslant z \leqslant 1$. Sketch S, then use the Divergence Theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} \,.\, \mathrm{d}\mathbf{S}$. Verify your result by calculating the surface integral directly. [*Answer:* 2π.]

6. By applying the Divergence Theorem to $\mathbf{k} \times \mathbf{A}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field defined in a volume V and on its surface, show that

$$\int_V \boldsymbol{\nabla} \times \mathbf{A}\, \mathrm{d}V = \int_{\partial V} \mathrm{d}\mathbf{S} \times \mathbf{A}.$$

Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $\mathbf{A}(\mathbf{x}) = (z, 0, 0)$. [*You should find that both integrals equal* $(0, abc, 0)$.]

* **7.** By applying Stokes' Theorem to the vector field $\mathbf{a} \times \mathbf{F}$ for any constant vector $\mathbf{a}$, or otherwise, show that for a vector field $\mathbf{F}(\mathbf{x})$,

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{F} = \int_S (\mathrm{d}\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{F}$$

where S is any suitable surface and $C = \partial S$. Verify this result when $\mathbf{F}(\mathbf{x}) = \mathbf{x}$ and C is the unit square in the (x, y)-plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$.

* **8.** Let f be a scalar field and $\mathbf{F}$ a vector field defined on a volume V. By applying the Divergence Theorem to suitable vector fields, prove that:

(i) If f is constant on ∂V then $\int_V \boldsymbol{\nabla} f \, \mathrm{d}V = \mathbf{0}$.

(ii) If $\boldsymbol{\nabla} \,.\, \mathbf{F} = 0$ in V and $\mathbf{F} \,.\, \mathbf{n} = 0$ on ∂V then $\int_V \mathbf{F}\, \mathrm{d}V = \mathbf{0}$.

Conservation Laws

9. For electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, define the quantities $U = \frac{1}{2}(\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2)$ and $\mathbf{P} = \mu_0^{-1}\mathbf{E} \times \mathbf{B}$. Use Maxwell's equations with $\mathbf{J} = \mathbf{0}$ to establish the conservation law $\partial U/\partial t + \boldsymbol{\nabla} \cdot \mathbf{P} = 0$.

If U has the interpretation of the energy density stored in the electromagnetic fields, what is the interpretation of the *Poynting vector* $\mathbf{P}$?

Laplace's and Poisson's Equations

10. The scalar field $\varphi = \varphi(r)$ depends only on $r = |\mathbf{x}|$ in $\mathbb{R}^3$. Use Cartesian coordinates and suffix notation to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\frac{\mathbf{x}}{r}, \quad \boldsymbol{\nabla}^2\varphi = \varphi''(r) + \frac{2}{r}\varphi'(r).$$

Verify this result using your expression for the Laplacian in spherical polar coordinates.

Solve the equation $\boldsymbol{\nabla}^2\varphi = 1$ in $r < a$, subject to $\varphi = 1$ on $r = a$.

What are the equivalent results in $\mathbb{R}^2$?

11. (i) Using Cartesian coordinates (x, y), find all solutions of Laplace's equation $\boldsymbol{\nabla}^2\phi = 0$ in two dimensions of the form $\phi(x, y) = f(x)e^{\alpha y}$ with α constant. Hence find a solution on the region $0 < x < a$, $y > 0$ with boundary conditions

$$\phi(0, y) = \phi(a, y) = 0, \quad \phi(x, 0) = \sin(\pi x/a), \quad \phi(x, y) \to 0 \text{ as } y \to \infty.$$

(ii) Using the formula for the Laplacian in plane polar coordinates (r, θ), verify that Laplace's equation in the plane has solutions of the form $\phi(r, \theta) = Ar^\alpha \cos\beta\theta$ so long as α and β are related appropriately. Hence find solutions on the following regions:

(a) $r < a$, subject to $\phi(a, \theta) = \cos\theta$;

(b) $r > a$, subject to $\phi(a, \theta) = \cos\theta$, $\phi(r, \theta) \to 0$ as $r \to \infty$;

(c) $a < r < b$, subject to $\partial\phi/\partial\mathbf{n} = -\cos 2\theta$ on $r = a$, $\phi(b, \theta) = 0$.

12. Let u be harmonic in a domain $\mathscr{D}$ and v be a smooth function satisfying $v = 0$ on $\partial\mathscr{D}$. Show that $\int_{\mathscr{D}} \boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}v \, \mathrm{d}V = 0$. Now if w is any smooth function in $\mathscr{D}$ with $w = u$ on $\partial\mathscr{D}$, show, by considering $v = w - u$, that $\int_{\mathscr{D}} |\boldsymbol{\nabla}w|^2 \, \mathrm{d}V \geqslant \int_{\mathscr{D}} |\boldsymbol{\nabla}u|^2 \, \mathrm{d}V$; that is to say, the solution of Laplace's equation minimises this integral.

* **13.** Show that at any point $\mathbf{a}$, a given harmonic function ϕ is equal to the average of its values in the ball $B_r(\mathbf{a}) = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| < r\}$, for any $r > 0$. By using this result for large r and considering $\boldsymbol{\nabla}\phi$, or otherwise, prove that if ϕ is bounded and harmonic on $\mathbb{R}^3$ then it is constant.

Uniqueness of Solutions to Laplace's and Poisson's Equations

14. Show that the solution to the following boundary value problem in a domain $\mathscr{D}$ is unique:

$$-\boldsymbol{\nabla}^2\phi + \phi = \rho \text{ in } \mathscr{D}, \quad \partial\phi/\partial\mathbf{n} = f \text{ on } \partial\mathscr{D}$$

where ρ and f are given functions on $\mathscr{D}$ and $\partial\mathscr{D}$ respectively.

15. Given a volume V with surface S, show that the solution to the boundary value problem

$$\boldsymbol{\nabla}^2\psi = 0 \text{ in } V, \quad g\frac{\partial\psi}{\partial\mathbf{n}} + \psi = f \text{ on } S$$

is unique, where $f(\mathbf{x})$ and $g(\mathbf{x})$ are specified functions on S obeying $g \geqslant 0$.

Show that uniqueness may not hold if the condition $g \geqslant 0$ is not enforced, by finding a nonzero solution to Laplace's equation on $|\mathbf{x}| \leqslant 1$ which satisfies $\psi = \partial\psi/\partial\mathbf{n}$ on $|\mathbf{x}| = 1$.