Mathematical Tripos Part IA — Lent 2021
Vector Calculus, Example Sheet 3 — Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let S be the *open* surface

$$1 - z = x^2 + y^2, \quad 0 \le z \le 1.$$

Use the divergence theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. Verify your result by calculating the area integral directly. *Hint: you should find that* $\mathrm{d}\mathbf{S} = (2\rho\cos\phi, 2\rho\sin\phi, 1)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$.

2. By applying the divergence theorem to the vector field $\mathbf{a} \times \mathbf{A}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \nabla \times \mathbf{A}\,\mathrm{d}V = \int_S \mathrm{d}\mathbf{S} \times \mathbf{A}$$

where $S = \partial V$. Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $\mathbf{A}(\mathbf{x}) = (z, 0, 0)$.

3. The scalar field $\varphi = \varphi(r)$ only depends on $r = |\mathbf{x}|$. Use Cartesian coordinates and suffix notation to show

$$\nabla\varphi = \varphi'(r)\frac{\mathbf{x}}{r}, \quad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\varphi'(r).$$

Verify this result using your expression for the Laplacian in spherical polar coordinates. Solve the equation

$$\begin{cases} \nabla^2\varphi = 1, & r < a \\ \varphi = 1, & r = a. \end{cases}$$

4. (a) Using Cartesian coordinates (x, y), find all solutions of Laplace's equation $\nabla^2\varphi = 0$ in two dimensions of the form $\varphi(x, y) = f(x)e^{\alpha y}$, with α constant. Hence find a solution on the region $0 < x < a$ and $y > 0$ with boundary conditions:

$$\varphi(0, y) = \varphi(a, y) = 0, \quad \varphi(x, 0) = \lambda\sin(\pi x/a), \quad \varphi(x, y) \to 0 \text{ as } y \to \infty.$$

(b) Using the formula for the Laplacian in plane polar coordinates (r, θ), verify that Laplace's equation in the plane has solutions of the form $\varphi(r, \theta) = Ar^\alpha\cos\beta\theta$, if α and β are related appropriately. Hence find solutions on the following regions, with the given boundary conditions (λ a constant):

(i) $r < a, \quad \varphi(a, \theta) = \lambda\cos\theta,$

(ii) $r > a, \quad \varphi(a, \theta) = \lambda\cos\theta, \quad \varphi(r, \theta) \to 0$ as $r \to \infty,$

(iii) $a < r < b, \quad \frac{\partial\varphi}{\partial\mathbf{n}}(a, \theta) = 0, \quad \varphi(b, \theta) = \lambda\cos 2\theta.$

5. Consider a complex valued function $f = \varphi(x, y) + \mathrm{i}\psi(x, y)$ satisfying $\partial f/\partial\bar{z} = 0$, where $\frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$. Show that $\nabla^2\varphi = \nabla^2\psi = 0$. Show also that a curve on which φ is constant is orthogonal to a curve on which ψ is constant, at a point where they intersect. Find φ and ψ when $f = ze^z$, $z = x + \mathrm{i}y$, and compare with question 8 on sheet 2.

6. Use Gauss' flux method to find the electric field $\mathbf{E} = \mathbf{E}(\mathbf{x})$ due to a spherically symmetric charge density

$$\rho(r) = \begin{cases} 0, & 0 \le r \le a \\ \rho_0 r/a, & a < r < b, \\ 0, & r \ge b. \end{cases}$$

Now find the electric potential $\phi = \phi(r)$ *directly* from Poisson's equation by writing down the general, spherically symmetric solution to Laplace's equation in each of the intervals $0 < r < a$, $a < r < b$ and $r > b$, and adding a particular integral where necessary. You should assume that ϕ and ϕ' are continuous at $r = a$ and $r = b$. Check this solution gives rise to the same electric field using $\mathbf{E} = -\nabla\phi$.

7. For the electric and magnetic fields $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ define the quantities

$$U = \frac{1}{2}\left(\varepsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right), \quad \mathbf{P} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}.$$

Use Maxwell's equations with $\mathbf{J} = 0$ to establish the conservation law $\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{P} = 0$.

8. Let φ and ψ be scalar functions. Using an integral theorem, establish *Green's second identity*

$$\int_V \left(\psi\nabla^2\varphi - \varphi\nabla^2\psi\right) \mathrm{d}V = \int_{\partial V}\left(\psi\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\psi}{\partial\mathbf{n}}\right)\mathrm{d}S.$$

9. Show that the solution to the following boundary value problem is unique

$$\begin{cases} -\nabla^2\varphi + \varphi = \rho, & \text{in } \Omega, \\ \partial\varphi/\partial\mathbf{n} = f, & \text{on } \partial\Omega. \end{cases}$$

10. Show that the solution to the following boundary value problem is unique

$$\begin{cases} \nabla^2\varphi = 0, & \text{in } \Omega, \\ (\partial\varphi/\partial\mathbf{n})g + \varphi = f, & \text{on } \partial\Omega, \end{cases}$$

assuming that $g(\mathbf{x}) \geq 0$ on $\partial\Omega$. Find a non-zero solution to Laplace's equation on $|\mathbf{x}| \leq 1$ which satisfies the boundary conditions above with $f = 0$ and $g = -1$ on $|\mathbf{x}| = 1$.

11. Let u be harmonic on Ω and v a smooth function that satisfies $v = 0$ on $\partial\Omega$. Show that

$$\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}V = 0.$$

Now if w is any function on Ω with $w = u$ on $\partial\Omega$, show, by considering $v = w - u$, that

$$\int_\Omega |\nabla w|^2\,\mathrm{d}V \geq \int_\Omega |\nabla u|^2\,\mathrm{d}V.$$

Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

12. For $\epsilon > 0$ define $\Phi_\epsilon(\mathbf{x}) = (|\mathbf{x}| + \epsilon)^{-1}$. Show that

$$\nabla^2\Phi_\epsilon(\mathbf{x}) = \frac{-2\epsilon}{|\mathbf{x}|\left(|\mathbf{x}| + \epsilon\right)^3}.$$

If φ is a scalar function that decays rapidly as $|\mathbf{x}| \to \infty$ and $\mathbf{a} \in \mathbf{R}^3$ is fixed, compute the limit

$$\lim_{\epsilon\to 0}\int_{\mathbf{R}^3}\varphi(\mathbf{x})\nabla^2\Phi_\epsilon(\mathbf{x} - \mathbf{a})\,\mathrm{d}V.$$

Deduce that $\nabla^2\left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{a}|}\right) = \delta(\mathbf{x} - \mathbf{a})$.

13. Show that a harmonic function φ at the point $\mathbf{a}$ is equal to the average of its values on the interior of the ball $B_r(\mathbf{a}) = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| < r\}$, for any $r > 0$. By considering $\nabla\varphi$ and the previous result for large r, or otherwise, prove that if φ is bounded and harmonic on $\mathbf{R}^3$ then it is constant.

14. (Harder) For a volume V with smooth boundary S, establish the identity $\mathrm{vol}(V) = \frac{1}{3}\int_S \mathbf{x}\cdot\mathrm{d}\mathbf{S}$. Suppose now that $V = V(t)$, and the velocity of a point $\mathbf{x} \in V$ is $\mathbf{v}(\mathbf{x})$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(V) = \int_S \mathbf{v}\cdot\mathrm{d}\mathbf{S}.$$

Using this result, or otherwise, obtain *Reynold's Transport Theorem* for a scalar function $\rho = \rho(\mathbf{x}, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\rho\,\mathrm{d}V = \int_{V(t)}\frac{\partial\rho}{\partial t}\,\mathrm{d}V + \int_{S(t)}\rho(\mathbf{v}\cdot\mathrm{d}\mathbf{S}).$$

Interpret this result.