Vector Calculus: Example Sheet 4

Part IA, Lent Term 2026

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Gauss' Flux Method

1. Use Gauss' flux method to find the electric field due to a spherically symmetric charge density

$$\rho(r) = \begin{cases} 0 & 0 \leqslant r \leqslant a, \\ \rho_0 r/a & a < r < b, \\ 0 & r \geqslant b. \end{cases}$$

Now find the electric potential $\phi(r)$ *directly* from Poisson's equation by writing down the general spherically symmetric solution to Laplace's equation in each of the three intervals, and adding a particular integral where necessary. You should assume that ϕ and ϕ' are continuous at $r = a$ and $r = b$. Check that this solution gives rise to the same electric field using $\mathbf{E} = -\boldsymbol{\nabla}\phi$.

Cartesian Tensors

2. The current $\mathbf{J}$ within an anisotropic conductive material due to an electric field $\mathbf{E}$ is given by $J_i = \sigma_{ij}E_j$, where σ_{ij} is the conductivity tensor. In a given Cartesian coordinate system,

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

3. Using the transformation law for a second-rank tensor T_{ij}, show that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are scalars, i.e., remain the same in all Cartesian coordinate systems. If T_{ij} is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

4. Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector and let $P_{ij} = \delta_{ij} - n_i n_j$. Show that P_{ij} is a tensor, and find its principal values and axes.

5. The velocity field $\mathbf{u}(\mathbf{x}, t)$ of an inviscid compressible gas obeys

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\,.\,(\rho\mathbf{u}) = 0 \quad \text{and} \quad \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\,\boldsymbol{\nabla})\mathbf{u}\Big) = -\boldsymbol{\nabla}p,$$

where $\rho(\mathbf{x}, t)$ is the density and $p(\mathbf{x}, t)$ is the pressure. Show that

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho u_i u_i) + \frac{\partial}{\partial x_i}(\tfrac{1}{2}\rho u_j u_j u_i + p u_i) = p\boldsymbol{\nabla}\,.\,\mathbf{u} \quad \text{and} \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial t_{ij}}{\partial x_j} = 0$$

for a suitable symmetric tensor t_{ij}, to be determined.

* What interpretation might be given to t_{ij}?

Symmetries of Tensors

6. (i) By first decomposing into symmetric and antisymmetric parts, show that an arbitrary second rank tensor T_{ij} can be written in the form

$$T_{ij} = \alpha\delta_{ij} + \epsilon_{ijk}\omega_k + D_{ij}$$

for some *scalar* α, *vector* $\boldsymbol{\omega}$ and symmetric traceless *tensor* D_{ij}.

(ii) Let $\mathbf{u}(\mathbf{x})$ be a vector field and let $T_{ij} = \partial u_i/\partial x_j$. Show that $\alpha = \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{u}$ and $\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{u}$. Find D_{ij} (the "deviatoric strain tensor") for the case $\mathbf{u}(\mathbf{x}) = (xy^2, yz^2, zx^2)$.

∗ Verify that $(0,0,1)$ is a principal axis of D_{ij} at the point $\mathbf{x} = (2,3,0)$, and find the others.

7. (i) A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

∗ (ii) A tensor of rank 4 satisfies $T_{jikl} = -T_{ijkl} = T_{ijlk}$ and $T_{ijij} = 0$. Show that $T_{ijkl} = \epsilon_{ijp}\epsilon_{klq}U_{pq}$ where $U_{pq} = -T_{rqrp}$. [*Hint: Try to generalise a technique pertaining to antisymmetric second rank tensors.*]

8. (i) The array of numbers T_{ijk} is such that $T_{ijk}C_{jk}$ is a vector for every second rank tensor C_{jk}. Show that T_{ijk} is a tensor.

(ii) The array T_{ijk} is such that $T_{ijk}A_{jk}$ is a vector for every antisymmetric second rank tensor A_{jk}. Show by counterexample that T_{ijk} need not be a tensor; but show also that $T_{ijk} - T_{ikj}$ must be.

9. (i) In a given basis $\mathscr{B}$ comprising three orthogonal axes, the physical shape and properties of a particular body are unchanged by rotations through an angle π about each of the axes. Show that any second rank tensor calculated for the body will be diagonal in $\mathscr{B}$, although the diagonal elements need not be equal.

(ii) A cuboid of uniform density and mass M has sides of length $2a$, $2b$ and $2c$. Find the inertia tensor about its centre, with respect to a coordinate system of your choice. Under what circumstances is the inertia tensor isotropic, and in that case what is the relationship between the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{L}$?

10. Evaluate the following integrals in $\mathbb{R}^3$, where $\gamma > 0$: $\displaystyle\int_{\mathbb{R}^3} r^{-3}e^{-\gamma r^2}x_i x_j \,\mathrm{d}V$; $\displaystyle\int_{r<1} r^{-5}e^{-\gamma r^2}x_i x_j x_k \,\mathrm{d}V$.

11. In linear elasticity, the symmetric second rank stress tensor σ_{ij} (representing the forces on an elastic medium) depends on the symmetric second rank strain tensor e_{kl} (representing the corresponding deformation) according to $\sigma_{ij} = c_{ijkl}e_{kl}$. Assuming that c_{ijkl} is symmetric in k and l, explain why it must be a fourth rank tensor. For an isotropic medium, use the most general possible form for c_{ijkl} (which you may quote) to show that

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

where λ and μ are scalars. Invert this equation to express e_{ij} in terms of σ_{ij}, assuming that $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of σ_{ij} and e_{ij} coincide.

∗ The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2}e_{ij}\sigma_{ij}$. Assuming that $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$, show that E is strictly positive for any non-zero strain e_{ij}.

∗ **12.** Let $T_{ijk\ldots}$ be an mth rank tensor on $\mathbb{R}^n$ (i.e., each of the m indices can take on n different values). How many independent components does $T_{ijk\ldots}$ have if it is (i) totally antisymmetric; or (ii) totally symmetric?

Tensor Calculus

13. If $\mathbf{u}(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second rank tensor field. If $\sigma_{ij}(\mathbf{x})$ is a tensor field, show that $\partial\sigma_{ij}/\partial x_j$ transforms as a vector field.