A7a Dynamics and Relativity: Examples Sheet 1 Lent 2024

Corrections and suggestions should be emailed to `S.A.Hartnoll@damtp.cam.ac.uk`*. Harder questions are starred and should be attempted after the others.*

1. In an inertial frame, a particle moves along a trajectory $\mathbf{x}(t)$ under the influence of a second heavy particle that is at rest at $\mathbf{x}_o$. The acceleration of the first particle is given by

$$m\ddot{\mathbf{x}}(t) = -k\frac{\mathbf{x}(t)-\mathbf{x}_o}{|\mathbf{x}(t)-\mathbf{x}_o|^d}\,. \tag{1}$$

Here m, k and d are constants. We may now consider this system in a different frame of reference. All positions in the new frame are related to positions in the old frame by $\mathbf{x}' = f(\mathbf{x}, t)$. Both frames use the same clock, so that $t' = t$. Consider, in particular, the transformations (a) – (c) listed below. In each case obtain an equation for $m\ddot{\mathbf{x}}'(t')$. Note that $'$ denotes the new frame and not a derivative.

(a) $\mathbf{x}' = R\mathbf{x}$, where the matrix R obeys $R^T R = 1$

(b) $\mathbf{x}' = \mathbf{x} + \mathbf{b}$, with $\mathbf{b}$ a constant vector

(c) $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$, with $\mathbf{v}$ a constant vector

How do your results relate to the Galilean invariance of physical laws?

Certain special forces may enjoy additional invariances. For what value of d is equation (1) invariant under the *Galilean spacetime rescaling* $\mathbf{x}' = \lambda\mathbf{x}$ and $t' = \lambda^2 t$, with λ a constant?

In lectures we obtained the acceleration due to gravity on Earth as

$$\ddot{\mathbf{x}}(t) = \mathbf{g}\,,$$

with $\mathbf{g}$ a fixed vector pointing downwards. Is this equation invariant under Galilean transformations? If not, why does is not violate the principle of Galilean relativity?

2. A particle of mass m at position $\mathbf{x}$ experiences a force

$$\mathbf{F} = \left(-\frac{a}{r^2} + \frac{2b}{r^3}\right)\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is the unit vector in the radial direction, the magnitude $r = |\mathbf{x}|$, and a and b are positive constants. Show, by finding a potential $V(r)$ such that $\mathbf{F} = -\nabla V$, that $\mathbf{F}$ is conservative. (Hint: you will need the result $\nabla r = \hat{\mathbf{x}}$).

Sketch the potential $V(r)$ and describe qualitatively the possible motions of the particle moving in the radial direction, considering different starting positions and speeds. If the particle starts at the point $r = 2b/a$, what is the minimum speed that the particle must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance R, much larger than the Earth's radius. Treating the Earth as a point of mass M, use dimensional analysis to show that the time T taken by the satellite to reach the Earth is given by

$$T = C\left(\frac{R^3}{GM}\right)^{\frac{1}{2}},$$

where G is the gravitational constant and C is a dimensionless constant. (The acceleration due to the Earth's gravitational field at a distance r from the centre of the Earth is GM/r^2).

What is the equation of energy conservation for the satellite? By solving this differential equation, show that $C = \pi/2\sqrt{2}$.

4. A long time ago, in a galaxy far far away, a perfectly spherical Death Star was constructed. Its surrounding force field caused a particle at position $\mathbf{x}$ relative to the Death Star to experience an acceleration

$$\ddot{\mathbf{x}} = \lambda\,\mathbf{x} \times \dot{\mathbf{x}}$$

where $\lambda > 0$ is a constant. Show that particles move in this field with constant speed. Show, moreover, that the magnitude of acceleration is also constant.

(a) A particle is projected *radially* with speed v from a point $\mathbf{x} = R\hat{\mathbf{x}}$ on the surface of the Death Star. Show that its trajectory is given by

$$\mathbf{x} = (vt + R)\hat{\mathbf{x}}$$

(b) By considering the second derivative of $\mathbf{x} \cdot \mathbf{x}$ show that, for any particle moving in the force field, the distance r to the centre of the Death Star is given by

$$r^2 = v^2\Big((t - t_0)^2 + t_1^2\Big)$$

where t_0 and $t_1 > 0$ are constants and v is the speed of the particle. Obtain an expression for $\mathbf{x} \cdot \dot{\mathbf{x}}$ and show that $|\ddot{\mathbf{x}}| = \lambda t_1 v^2$.

5. A particle of mass m, charge q and position $\mathbf{x}$ moves in a constant, uniform field $\mathbf{B}$ which points in a horizontal direction. The particle is also under the influence of gravity, $\mathbf{g}$, acting vertically downwards. Write down the equation of motion and show that it is invariant under translations $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0$. Obtain

$$\dot{\mathbf{x}} = \alpha\mathbf{x} \times \mathbf{n} + \mathbf{g}t + \mathbf{a}$$

where $\alpha = qB/m$, $\mathbf{n}$ is a unit vector in the direction of $\mathbf{B}$ and $\mathbf{a}$ is a constant vector. Show that, with a suitable choice of origin, $\mathbf{a}$ can be written in the form $\mathbf{a} = a\mathbf{n}$.

By choosing suitable axes, show that the particle undergoes a helical motion with a constant horizontal drift.

Suppose that you now wish to eliminate the drift by imposing a uniform electric field $\mathbf{E}$ Determine the direction and magnitude of $\mathbf{E}$.

6. At time $t = 0$, an insect of mass m jumps from a point O on the ground with velocity $\mathbf{v}$, while a wind blows with velocity $\mathbf{u}$. The gravitational acceleration is $\mathbf{g}$ and the air exerts a retarding force on the insect equal to mk times the velocity of the wind *relative to the insect*.

   (a) Show that the path of the insect is given by

   $$\mathbf{x} = (\mathbf{u} + \mathbf{g}/k)t + \frac{1 - e^{-kt}}{k}(\mathbf{v} - \mathbf{u} - \mathbf{g}/k)$$

   (b) In the case where the insect jumps vertically in a horizontal wind, show that the time T that elapses before it returns to earth satisfies

   $$(1 - e^{-kT}) = \frac{kT}{1 + \gamma}$$

   where $\gamma = kv/g$. Find an expression for the range R in terms of γ, u and T. (Here $v = |\mathbf{v}|$, $g = |\mathbf{g}|$, and $u = |\mathbf{u}|$.) Discuss the limits of small and large γ.

7. A ball of mass m moves in a resisting medium that produces a friction force of magnitude kv^2, where v is the ball's speed. If the ball is projected vertically upwards with initial speed u, show by dimensional analysis that when the ball returns to its point of projection, its speed w can be written in the form

   $$w = uf(\lambda)\,,$$

   where $\lambda = ku^2/mg$. Integrate the equations of motion to show that $f(\lambda) = (1+\lambda)^{-1/2}$. Discuss the physics in the two extremes $\lambda \gg 1$, and $\lambda \ll 1$.

   [*Hint: Thinking about v as a function of time may not be the easiest approach*]

8. (∗) The temperature $\theta(x, t)$ in a very long rod is governed by the one-dimensional diffusion equation

   $$\frac{\partial\theta}{\partial t} = D\frac{\partial^2\theta}{\partial x^2}$$

   where D is a constant (the *thermal diffusivity* of the rod). At time $t = 0$, the point $x = 0$ is heated to a high temperature. At all later times, the conservation of heat energy implies that

   $$Q = \int_{-\infty}^{\infty} \theta(x, t)dx$$

   is constant. Use dimensional analysis to show that $\theta(x, t)$ can be written in the form

   $$\theta(x, t) = \frac{Q}{\sqrt{Dt}}F(z)$$

   where $z = x/\sqrt{Dt}$ and show further that

   $$\frac{d^2F}{dz^2} + \frac{z}{2}\frac{dF}{dz} + \frac{1}{2}F = 0$$

   Integrate this equation once directly to obtain a first order differential equation (you may assume that $zF(z) \to 0$ and $dF(z)/dz \to 0$ as $z \to \infty$), and hence show that

   $$\theta(x, t) = \frac{Q}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}.$$