

Example Sheet 2

[Example Sheet 2 is based on (approximately) Lectures 8-12 of the course.]

1. A particle moves in a fixed plane and its position vector at time t is \mathbf{r} . Let (r, θ) be plane polar coordinates and let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the directions of increasing r and increasing θ , respectively. Show that

$$\dot{\mathbf{r}} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}}.$$

The particle moves outwards with speed $v(t)$ on the equiangular spiral $r = a \exp(\theta \cot \alpha)$, where a and α are constants, with $0 < \alpha < \frac{1}{2}\pi$. Show that

$$v \sin \alpha = r \dot{\theta},$$

and hence that

$$\dot{\mathbf{r}} = v \cos \alpha \hat{\mathbf{r}} + v \sin \alpha \hat{\boldsymbol{\theta}}.$$

Give an expression for $\ddot{\mathbf{r}}$ and show that $|\ddot{\mathbf{r}}|^2 = \dot{v}^2 + v^2 \dot{\theta}^2$.

(*) If $\dot{\theta}$ takes a constant value ω , show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

2. In each part of this question, the particles move in a gravitational potential $\Phi_g(r) = -k/r$ with $k > 0$. You should answer each part using *only* energy and angular momentum conservation, and (for the circular orbits) the radial component of the equation of motion. Where needed recall that for a non-circular orbit *periapsis* is the closest point on the orbit to the focus) and *apoapsis* is the furthest point.

(a) Show that the radius, R , of the orbit of a satellite in geostationary orbit (in the equatorial plane) is approximately $(28)^{-2/3} R_M$, where R_M is the radius of the Moon's orbit around the Earth.

(b) One particle moves in a parabolic orbit and another particle moves in a circular orbit. Show that if they pass through the same point then the ratio of their speeds at this point is $\sqrt{2}$. For a satellite orbiting the Earth in a circular orbit, what is the relationship between its orbital speed and its escape velocity?

If, instead of passing through the same point, the particles have the same angular momentum per unit mass, show that the periapsis distance of the parabola is half the radius of the circle.

- (c) A particle moves with angular momentum h per unit mass in an ellipse, for which the distances from the focus to the periapsis and apoapsis are p and q , respectively. Show that

$$h^2 \left(\frac{1}{p} + \frac{1}{q} \right) = 2k.$$

Show also that the speed V of the particle at the periapsis is related to the speed v of a particle moving in a circular orbit of radius p by $(1 + p/q)V^2 = 2v^2$.

- (d) A particle P is initially at a very large distance from the origin moving with speed v on a trajectory that, in the absence of any force, would be a straight line for which the shortest distance from the origin is b . The shortest distance between P 's actual trajectory and the origin is d . Show that $2kd = v^2(b^2 - d^2)$.

3. A particle of unit mass moves with speed v in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are μ/r^2 towards the Sun and kv opposing the motion, where μ and k are constants. Write down the vector equation of motion and show that the vector \mathbf{H} , defined by

$$\mathbf{H} = e^{kt} \mathbf{r} \times \dot{\mathbf{r}},$$

is constant. Deduce that the particle moves in a plane through the origin. Establish the equations

$$r^2 \dot{\theta} = h e^{-kt} \quad \text{and} \quad \mu r = h^2 e^{-2kt} - r^3 (\ddot{r} + k\dot{r}),$$

where r and θ are plane polar coordinates centred on the Sun and h is a constant. Show that, when $k = 0$, a circular orbit of radius a exists for any value of a , and find its angular frequency ω in terms of a and μ .

When $k/\omega \ll 1$, r varies so slowly that \dot{r} and \ddot{r} may be neglected in the above equations. Verify that in this case an approximate solution is

$$r = a e^{-2kt}, \quad \dot{\theta} = \omega e^{3kt}.$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

4. A particle P of unit mass moves in a plane under a central force

$$F(r) = -\frac{\lambda}{r^3} - \frac{\mu}{r^2},$$

where λ and μ are positive constants. Write down the differential equation satisfied by $u(\theta)$, where $u = 1/r$.

Given that P is projected with speed V from the point $r = r_0$, $\theta = 0$ in the direction perpendicular to OP , find the equation of the orbit under the assumptions

$$\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda.$$

Explain the significance of these inequalities. Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

$$\pi \left(1 - \frac{\lambda}{V^2 r_0^2} \right)^{-1/2}.$$

Under what condition is the orbit a closed curve?

5. For a particle of mass m subject to an inverse-square force given by $\mathbf{F} = -mk\hat{\mathbf{r}}/r^2$, the vectors \mathbf{h} and \mathbf{e} are defined by

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}, \quad \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{k} - \frac{\mathbf{r}}{r}.$$

Show that \mathbf{h} is constant and deduce that the particle moves in a plane through the origin.

The vector \mathbf{e} is known as the *eccentricity vector* or *Laplace–Runge–Lenz vector*. Show that it too is constant and that

$$er \cos \theta = \frac{h^2}{k} - r,$$

where $e = |\mathbf{e}|$, $h = |\mathbf{h}|$ and θ is the angle between \mathbf{r} and \mathbf{e} . Deduce that the orbit is a conic section.

6. (**Additional question on orbits – attempt only after 1-5.**) A particle P of mass m moves under the influence of a central force of magnitude mk/r^3 directed towards a fixed point O . Initially $r = a$ and P has velocity v perpendicular to OP , where $v^2 < k/a^2$. Use the differential equation for the shape of the orbit to prove that P spirals in towards O (you should give the geometric equation of the spiral). Show also that it reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2 v^2}}.$$

7. (**Additional question on orbits – attempt only after 1-5.**) A particle of mass m moves in a circular orbit of radius R under the influence of an attractive central force of magnitude $F(r)$. Obtain an equation relating R , $F(R)$, m and the orbital angular momentum per unit mass, h .

The particle experiences a very small radial perturbation of the form $u(\theta) = U + \epsilon(\theta)$, where $u = 1/r$ and $U = 1/R$. The orbital angular momentum is not affected. Obtain the equation for $\epsilon''(\theta)$. Given that the subsequent orbit is closed, show that

$$\frac{RF'(R)}{F(R)} = \beta^2 - 3,$$

where β is a rational number. Deduce that, if β is independent of R , then $F(r)$ is of the form Ar^α , where α is rational and greater than -3 .

8. In each part of this question, use ω for the angular speed of the Earth, assume that events take place at latitude θ in the northern hemisphere and ignore centrifugal forces.

- (a) Are bath-plug vortices in the northern hemisphere likely, on average, to be clockwise or anticlockwise?
- (b) A straight river flows with speed v in a direction α degrees east of north. Show that the effect of the Coriolis force is to erode the right bank. Calculate the magnitude of the force.
- (c) A plumb line is attached to the ceiling inside one of the carriages of a train and hangs down freely, at rest relative to the train. When the train is travelling at speed V in the north-easterly direction the plumb line hangs at an angle ϕ to the direction in which it hangs when the train is at rest. Ignoring centrifugal forces, show that $\phi \approx (2\omega V \sin \theta)/g$. Under what circumstances can ignoring the centrifugal force be justified?

9. A bullet of mass m is fired from a point \mathbf{r}_0 with velocity \mathbf{u} in a frame that rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. The bullet is subject to a gravitational force $m\mathbf{g}$ which is constant in the rotating frame. Using the vector equation of motion and neglecting terms of order $|\boldsymbol{\omega}|^2$, show that the bullet's position vector measured in the rotating frame is approximately

$$\mathbf{r}_0 + \mathbf{u}t + \left(\frac{1}{2}\mathbf{g} - \boldsymbol{\omega} \times \mathbf{u}\right)t^2 + \frac{1}{3}(\mathbf{g} \times \boldsymbol{\omega})t^3$$

at time t . Suppose that the bullet is projected from sea level on the Earth at latitude θ in the northern hemisphere, at an angle $\pi/4$ from the upward vertical and in a northward direction. Show that when the particle returns to sea level (neglecting the curvature of the Earth's surface), it has been deflected to the east by an amount approximately equal to

$$\frac{\sqrt{2}\omega|\mathbf{u}|^3}{3g^2}(3\sin\theta - \cos\theta),$$

where ω is the angular speed of the Earth. Evaluate the approximate size of this deflection at latitude 52° N for $|\mathbf{u}| = 1000 \text{ m s}^{-1}$.

Please send any comments and corrections to pjh1@cam.ac.uk