Mathematical Tripos Part IB H.S. Reall

ELECTROMAGNETISM Examples 1 Lent 2021

1. Suppose that the current density is $\mathbf{J}(t,\mathbf{x}) = C\,\mathbf{x}\,e^{-at\mathbf{x}^2}$, where C and a are constants. Show that the charge conservation equation can be satisfied by writing the charge density in the form $\rho = [f(\mathbf{x}) + t\,g(\mathbf{x})]\,e^{-at\mathbf{x}^2}$ where f and g are to be determined.

2. In a fluid environment, charge undergoes *diffusion*. This is described empirically by *Fick's law* $\mathbf{J} = -D\nabla\rho$ where D is called the diffusion coefficient. Show that ρ obeys the heat equation. Show that this is solved by a spreading Gaussian of the form,

$$\rho(t,\mathbf{x}) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}} \exp\left(-\frac{\mathbf{x}^2}{4D(t-t_0)+a^2}\right),$$

where a, t_0 and ρ_0 constants.

3. Show that the motion of a particle of charge q and mass m in a constant magnetic field $(0,0,B)$ consists of motion with constant velocity in the z-direction combined with circular motion with angular frequency qB/m in the xy plane. Thus the trajectory of a charged particle in a constant magnetic field is a helix. How is the geometry of the helix related to the initial velocity of the particle?

4. Roughly sketch the electric field lines (including arrows to denote sense) and equipotentials (surfaces of constant Φ) for the following systems of point charges (i) A single charge $+q$; (ii) two charges $+q$ separated by a distance $2a$; (iii) two charges $\pm q$ separated by a distance $2a$.

5. A charge density is given by $\rho = \rho_0\,e^{-k|z|}$ with ρ_0 and k positive constants. This is invariant under translations in the xy-plane, rotations about the z-axis, and reflections in the xy-plane, so we assume the electric field has the same symmetries. Show that this implies $\mathbf{E} = (0,0,E(z))$ with $E(z) = -E(-z)$. Use Gauss' law to show that, for $z > 0$,

$$E(z) = \frac{\rho_0}{\epsilon_0 k}\left(1 - e^{-kz}\right).$$

6. (a) Use Gauss's law to obtain the electric field due to a uniform charge density ρ occupying the region $a < r < b$, with r the radial distance from the origin. (b) Show that in the limit $b \to a,\ \rho \to \infty$ with $(b-a)\rho$ fixed, the electric field suffers the expected discontinuity due to surface charge.

7. A circular disc of radius a has uniform surface charge density σ. Compute the potential at a point on the axis of symmetry at distance z from the centre. Compute the electric field at this point. Find the discontinuity in the normal electric field at the centre of the disc. Show that, far along the axis of symmetry, the electric field looks approximately like that of a charged point particle.

8. A capacitor consists of two conductors occupying the regions $a < r < b$ and $c < r < d$ in spherical polar coordinates, where $b < c$. Calculate its capacitance.

9. (a) A uniqueness theorem for Poisson's equation. Let V be a connected region with boundary S. Consider two solutions Φ_1 and Φ_2 for Poisson's equation (with the same RHS) in V. Define $\Psi = \Phi_1 - \Phi_2$. Show that

$$\int_V (\nabla\Psi)^2 dV = \int_S \Psi\nabla\Psi \cdot d\mathbf{S}$$

Now assume that Φ_1 and Φ_2 are constant on S. Use Gauss' law to deduce that Ψ is constant throughout V (so the two solutions give the same electric field).

(b) Consider a conductor with a cavity V inside it. Assume that the charge density vanishes in V. Use the result of (a) to prove that $\mathbf{E} = 0$ in V. Thus the conductor *screens* the cavity from the effects of any electric field outside the conductor.

(c) Consider a conductor occupying a finite region V', possibly containing cavities. Let S' be a surface lying infinitesimally outside V'. Show that the electric field outside S' depends only on the total charge enclosed by S', i.e., any surface charge on the conductor plus charge inside any cavities. [*Hint*: apply the method of (a) to the infinite region outside the conductor.]

(d) Consider a generalization of question (8) where we now allow the two conductors to have arbitrary shape, but still with spherical topology. Explain why (b) and (c) imply that $\mathbf{E}$ vanishes inside the inner conductor and outside the outer conductor. What does this imply about the surface charges?

10. ($\star$) A spherical conducting shell has radius R. The shell is grounded (i.e. has zero potential). A charge q is placed inside the shell at at point $\mathbf{x}_0 = (0, 0, d)$ from the centre, with $d < R$. Show that the potential inside the shell can be determined by placing an appropriate image charge outside the shell at $\mathbf{x}_0' = (0, 0, R^2/d)$. Show that the induced surface charge on the shell is

$$\sigma = -\frac{q}{4\pi}\frac{R^2 - d^2}{R(R^2 - 2dR\cos\theta + d^2)^{3/2}}$$

where θ is the angle between the point on the shell and the z-axis.

11. (a) Show that, far from a charge distribution $\rho(\mathbf{x})$ localised in region V, the potential takes the form

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\frac{Q_{ij}x_i x_j}{r^5} + \ldots\right)$$

where Q is the total charge, $\mathbf{p}$ is the dipole moment, Q_{ij} is the quadrupole moment tensor and $r = |\mathbf{x}|$.

(b) Compute Q, $\mathbf{p}$ and Q_{ij} for: (i) two charges, $+q$ and $-q$, at points $(0, 0, 0)$ and $(d, 0, 0)$ respectively; (ii) two charges $+q$ and two charges $-q$ placed on the corners of a square, with sides of length d, such that every charge has an opposite charge for each of its neighbours; (iii) four charges $+q$ and four charges $-q$ placed on the corners of a cube, with sides of length d, such that every charge has an opposite charge from each of its neighbours.

12. Show that the force and torque on a point electric dipole at position $\mathbf{x}$ in an electrostatic field are $\mathbf{F} = \mathbf{p}\cdot\nabla\mathbf{E}$ and $\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} + \mathbf{x}\times\mathbf{F}$. Deduce that the potential energy of the dipole is $-\mathbf{p}\cdot\mathbf{E}$. Hence show that the electrostatic energy of a pair of dipoles is

$$E = \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - \frac{3\mathbf{p}_1\cdot(\mathbf{x}_1 - \mathbf{x}_2)\,\mathbf{p}_2\cdot(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^5}\right]$$