Mathematical Tripos Part IB H.S. Reall

ELECTROMAGNETISM Examples 2 Lent 2021

1. A constant magnetic field points along the z-axis: $\mathbf{B} = B\mathbf{e}_z$. Verify that each of the following vector potentials satisfies $\mathbf{B} = \nabla \times \mathbf{A}$: (a) $\mathbf{A} = xB\mathbf{e}_y$, (b) $\mathbf{A} = \frac{B}{2}(x\mathbf{e}_y - y\mathbf{e}_x)$, (c) In cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}rB\mathbf{e}_\phi$, (d) in spherical polar coordinates, $\mathbf{A} = \frac{1}{2}r\sin\theta\, B\mathbf{e}_\phi$.

2. (a) A cylindrical conductor of radius a, with axis along the z-axis, carries a uniform current density $\mathbf{J} = J\mathbf{e}_z$. Show that the magnetic field within the conductor is given, in cylindrical polar coordinates, by $\mathbf{B} = \frac{1}{2}\mu_0 Jr\mathbf{e}_\phi$.

   (b) A steady current I flows in the z-direction uniformly in the the region between the cylinders $x^2 + y^2 = a^2$ and $(x + d)^2 + y^2 = b^2$, where $0 < d < (b - a)$. Use the superposition principle to show that the associated magnetic field $\mathbf{B}$ throughout the region $x^2 + y^2 < a^2$ is given by

$$\mathbf{B} = \frac{\mu_0 Id}{2\pi(b^2 - a^2)}\mathbf{e}_y$$

3. Use the Biot-Savart law to determine the magnetic field

   (a) around an infinite, straight wire carrying current I,

   (b) at the centre of a square loop of wire, with sides of length a, carrying current I,

   (c) at the point $(0, 0, z)$ above a circular loop of wire of radius a, lying in the (x, y) plane, with centre at the origin, carrying current I.

4. (a) A steady current I_1 flows around a closed loop C_1. Use the Biot-Savart law to show that the force exerted on this loop by the magnetic field produced by a second loop C_2 carrying current I_2 is

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi}I_1 I_2 \int_{C_1}\int_{C_2} d\mathbf{x}_1 \times \left(d\mathbf{x}_2 \times \frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3}\right)$$

   (b) Show how to write this in a form which satisfies Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

5. A surface current experiences a Lorentz force from the *average* magnetic field on either side of the surface. A wire carrying current I winds N times per unit length to form an infinite cylindrical solenoid. Show that the force per unit area on the cylinder is $\frac{\mu_0 I^2 N^2}{2}\mathbf{n}$ where $\mathbf{n}$ is the outward unit normal.

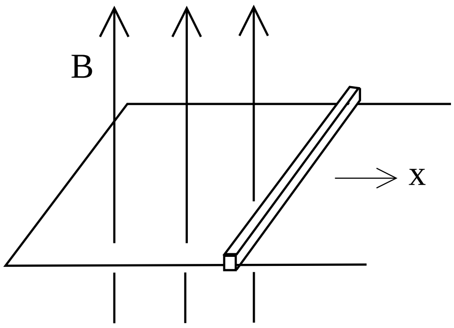
6. (a) An atom has magnetic dipole moment $\mathbf{m}$. It is arranged experimentally that $\mathbf{m}$ is anti-parallel to the local magnetic field. Show that the potential energy V_{dipole} of the atom is minimized when $|\mathbf{B}|$ is minimized.

   (b) Two infinite straight wires lie at $x = \pm a$ and $y = 0$ and carry current I in the z-direction. There is also an external magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. By expanding in x and y, show that the potential energy of the atom has a local minimum at $x = y = 0$ and determine the frequency of small oscillations about this minimum when $B_0 \gg (\mu_0 I)/(\pi a)$. (The result will depend on the mass M of the atom. This is an example of *magnetic trapping*.)

7. (a) A current creates time-dependent electric and magnetic fields, which in cylindrical polar coordinates are given by $\mathbf{E} = e^{-t}\mathbf{e}_\phi$ and $\mathbf{B} = r^{-1}e^{-t}\mathbf{e}_z$. Verify that these satisfy Maxwell's equations for vanishing charge density and determine the current density.

   (b) Let $S(t)$ be a disc in the plane $z = 0$ with radius $R(t) = 1 + t$ and centred on the origin. Let $C(t)$ be the circular boundary of $S(t)$. Evaluate the magnetic flux through $S(t)$ and the emf around $C(t)$. Verify that your results satisfy Faraday's law of induction.

8. A horizontal, rectangular circuit, shown in the figure, has a sliding bar of mass m and length L which moves, without friction, in the x-direction. The sides of the rectangle have lengths L and $x(t)$. The bar has resistance R and the other edges of the circuit have negligible resistance. A uniform vertical magnetic field $\mathbf{B} = B(t)\mathbf{e}_z$ is applied for time $t > 0$.



   Obtain an expression for the current $I(t)$ that flows around the circuit. Also obtain an expression for the Lorentz force $\mathbf{F}$ on the bar in terms of $B(t)$ and $I(t)$. Hence show that $x(t)$ must satisfy the differential equation

$$\frac{d^2x}{dt^2} = -\frac{BL^2}{MR}\frac{d}{dt}(Bx)$$

   Solve this equation for the case of constant $B(t)$. Sketch the solution $x(t)$ for $\dot{x}(0) > 0$.

   [In this question, and the following question, you may assume that the effect on the magnetic field due to any current flow is negligible compared to the background $\mathbf{B}$.]

9. A vector potential is given by $\mathbf{A} = \frac{1}{2}Brz\mathbf{e}_\phi$ in cylindrical polar coordinates, where B is a constant.

   (a) Compute the magnetic field $\mathbf{B}$.

   (b) A thin conducting wire of resistance R is formed into a circular loop of radius a. The loop lies in the plane $z = z(t)$ with its centre on the z-axis. Find the induced current in the loop.

   (c) Compute the force exerted on the loop by the magnetic field. To overcome this, an equal and opposite force is applied to the loop. Show that the work done per unit time by this force is equal to the rate of dissipation of energy due to the resistance in the loop.

10. A steady current I flows along a cylindrical conductor of constant circular cross-section and uniform conductivity σ. Show, using the relevant equations for $\mathbf{E}$ and $\mathbf{J}$, that the current is distributed uniformly across the cross-section of the cylinder, and calculate the electric and magnetic fields just outside the surface of the cylinder.

    Verify that the integral of the Poynting vector over unit length of the surface is equal to the rate per unit length of dissipation of electrical energy as heat.