Mathematical Tripos Part IB: Lent 2021

Numerical Analysis – Exercise Sheet 2[1]

1. Let $h = 1/M$, where $M \ge 1$ is an integer, and let Euler's method be applied to calculate the estimates $\{\boldsymbol{y}_n\}_{n=1,2,\ldots,M}$ of $\boldsymbol{y}(nh)$ for each of the differential equations

$$y' = -\frac{y}{1+t} \quad \text{and} \quad y' = \frac{2y}{1+t}, \qquad 0 \le t \le 1,$$

starting with $y_0 = y(0) = 1$ in both cases. By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for y_n, $n = 1, 2, \ldots, M$, which should be free from summations and products of n terms. Hence deduce the exact solutions of the equations from the limit $h \to 0$. Verify that the magnitude of the errors $y_n - y(nh)$, $n = 1, 2, \ldots, M$, is at most $\mathcal{O}(h)$.

2. Assuming that $\boldsymbol{f}$ satisfies the Lipschitz condition and possesses a bounded third derivative in $[0, t^*]$, apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$$

converges and that $\|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| \le ch^2$ for some $c > 0$ and all n such that $0 \le nh \le t^*$.

3. The s-step Adams–Bashforth method is of order s and has the form

$$\boldsymbol{y}_{n+s} = \boldsymbol{y}_{n+s-1} + h\sum_{j=0}^{s-1} \sigma_j \boldsymbol{f}(t_{n+j}, \boldsymbol{y}_{n+j}).$$

Calculate the actual values of the coefficients in the case $s = 3$

4. By solving a three-term recurrence relation, calculate analytically the sequence of values $\{\boldsymbol{y}_n : n = 2, 3, 4, \ldots\}$ that is generated by the *explicit midpoint rule*

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}),$$

when it is applied to the ODE $y' = -y$, $t \ge 0$. Starting from the values $y_0 = 1$ and $y_1 = 1 - h$, show that the sequence diverges as $n \to \infty$ for *all* $h > 0$. Recall, however, that order ≥ 1, the root condition and suitable starting conditions imply convergence in a *finite* interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.
Hint: In the last part, relate the roots of the recurrence relation to $\pm e^{\mp h} + \mathcal{O}(h^3)$.

5. Show that the multistep method

$$\sum_{j=0}^{3} \rho_j \boldsymbol{y}_{n+j} = h\sum_{j=0}^{2} \sigma_j \boldsymbol{f}(t_{n+j}, \boldsymbol{y}_{n+j}), \qquad \text{where } \rho_3 = 1 \text{ (as in lectures)},$$

[1]Corrections and suggestions should be emailed to `cbs31@cam.ac.uk`.

is fourth order only if the conditions $\rho_0 + \rho_2 = 8$ and $\rho_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition

6. An s-stage explicit Runge–Kutta method of order s with constant step size $h > 0$ is applied to the differential equation $y' = \lambda y$, $t \geq 0$. Prove the identity

$$y_n = \left[\sum_{l=0}^{s} \frac{1}{l!}(h\lambda)^l\right]^n y_0, \qquad n = 0, 1, 2, \ldots.$$

7. The following four-stage Runge–Kutta method has order four,

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}(t_n, \boldsymbol{y}_n) \\
\boldsymbol{k}_2 &= \boldsymbol{f}(t_n + \tfrac{1}{3}h, \boldsymbol{y}_n + \tfrac{1}{3}h\boldsymbol{k}_1) \\
\boldsymbol{k}_3 &= \boldsymbol{f}(t_n + \tfrac{2}{3}h, \boldsymbol{y}_n - \tfrac{1}{3}h\boldsymbol{k}_1 + h\boldsymbol{k}_2) \\
\boldsymbol{k}_4 &= \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1 - h\boldsymbol{k}_2 + h\boldsymbol{k}_3) \\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h(\tfrac{1}{8}\boldsymbol{k}_1 + \tfrac{3}{8}\boldsymbol{k}_2 + \tfrac{3}{8}\boldsymbol{k}_3 + \tfrac{1}{8}\boldsymbol{k}_4).
\end{aligned}$$

By considering the equation $y' = y$, show that the order is at most four. Then, *for scalar functions,* prove that the order is at least four in the easy case when f is independent of y, and that the order is at least three in the relatively easy case when f is independent of t. *[Comment: do not derive all of the (gory) details when $f(t, y)$ depends on both t and $\boldsymbol{y}$.]*

8. Find $\mathcal{D} \cap \mathbb{R}$, the intersection of the linear stability domain $\mathcal{D}$ with the real axis, for the following methods:

(1) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$
(2) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$
(3) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})$
(4) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_{n+1} + \frac{1}{2}h[3\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) - \boldsymbol{f}(t_n, \boldsymbol{y}_n)]$
(5) The RK method $\boldsymbol{k}_1 = \boldsymbol{f}(t_n, \boldsymbol{y}_n)$, $\boldsymbol{k}_2 = \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1)$, $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h(\boldsymbol{k}_1 + \boldsymbol{k}_2)$.

9. Show that, if z is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$\boldsymbol{y}_{n+2} - \tfrac{4}{3}\boldsymbol{y}_{n+1} + \tfrac{1}{3}\boldsymbol{y}_n = \tfrac{2}{3}hf(t_{n+2}, \boldsymbol{y}_{n+2})$$

then the real part of z is positive. Thus deduce that this method is A-stable.

10. The (stiff) differential equation

$$y'(t) = -10^4(y - t^{-1}) - t^{-2}, \qquad t \geq 1, \qquad y(1) = 1,$$

has the analytic solution $y(t) = t^{-1}$, $t \geq 1$. Let it be solved numerically by Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n)$ and the backward Euler method $y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$, where $h_n = t_{n+1} - t_n$ is allowed to depend on n and to be different in the two cases. Suppose that, for any $t_n \geq 1$, we have $|y_n - y(t_n)| \leq 10^{-6}$, and that we require $|y_{n+1} - y(t_{n+1})| \leq 10^{-6}$. Show that Euler's method can fail if $h_n = 2 \times 10^{-4}$, but that the backward Euler method

always succeeds if $h_n \le 10^{-2} t_n t_{n+1}^2$.
Hint: Find relations between $y_{n+1} - y(t_{n+1})$ and $y_n - y(t_n)$ for general y_n and t_n.

11. This question concerns the predictor-corrector pair

$$\begin{aligned} \boldsymbol{y}_{n+3}^{\mathrm{P}} &= -\tfrac{1}{2}\boldsymbol{y}_n + 3\boldsymbol{y}_{n+1} - \tfrac{3}{2}\boldsymbol{y}_{n+2} + 3h\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2}), \\ \boldsymbol{y}_{n+3}^{\mathrm{C}} &= \tfrac{1}{11}[2\boldsymbol{y}_n - 9\boldsymbol{y}_{n+1} + 18\boldsymbol{y}_{n+2} + 6h\boldsymbol{f}(t_{n+3}, \boldsymbol{y}_{n+3})]. \end{aligned}$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value $\frac{6}{17}|\boldsymbol{y}_{n+3}^{\mathrm{P}} - \boldsymbol{y}_{n+3}^{\mathrm{C}}|$.

12. Let $u(x)$, $0 \le x \le 1$, be a six-times differentiable function that satisfies the ODE $u''(x) = f(x)$, $0 \le x \le 1$, $u(0)$ and $u(1)$ being given. Further, we let $x_m = mh = m/M$, $m = 0, 1, \ldots, M$, for some positive integer M, and calculate the estimates $u_m \approx u(x_m)$, $m = 1, 2, \ldots, M-1$, by solving the difference equation

$$u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) + \alpha h^2[f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \qquad m = 1, 2, \ldots, M-1,$$

where $u_0 = u(0)$, $u_M = u(1)$, and α is a positive parameter. Show that there exists a choice of α such that the local truncation error of the difference equation is $\mathcal{O}(h^6)$.