B6a **Variational Principles: Example Sheet 1** Easter term 2026

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1. At how many points does the function

$$\phi(x_1, x_2, x_3) = \tfrac{1}{4}(x_1^4 + x_2^4 + x_3^4) - x_2x_3 - x_3x_1 - x_1x_2$$

take its minimum value? Show that this least value is -3. Show that there is one other stationary point, at which $\phi = 0$; by considering the eigenvalues of the Hessian, show that it is a saddle point. Show also that the surface $\phi = 0$ near this point is tangent to (and hence "looks like") a double cone of semi-angle $\arctan\sqrt{2}$ [*Hint: Use the fact that there exist coordinates* (x, y, z) *such that* $\phi \approx \frac{1}{2}(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) \equiv Q(x, y, z)$ *near any stationary point, where* $(\lambda_1, \lambda_2, \lambda_3)$ *are the eigenvalues of the Hessian at that point, and examine the nature of the surface defined by* $Q(x, y, z) = 0$.]

2. Show that:

(i) x^2/y is convex on the upper half plane $(x, y) : y > 0$.

(ii) the function $F(x, y) = yf(x/y)$ (called the "perspective" of f) is convex on $(x, y) : y > 0$ if $f(x)$ is convex [*Hint: after introducing* $t \in (0, 1)$ *use the new variable* $s = \frac{ty'}{(1-t)y+ty'}$]. Now, assuming f to be twice differentiable, verify convexity of F by computing its Hessian matrix.

3. Find the Legendre transform of $f(x) = e^x$, (giving its domain also). Find the Legendre transform of $f(x) = a^{-1}x^a, a > 1$ defined on $x > 0$, and hence deduce Young's inequality

$$xy \;\le\; \frac{x^a}{a} + \frac{y^b}{b}\,, \qquad \frac{1}{a} + \frac{1}{b} = 1\,.$$

4. An isolated thermodynamic system in equilibrium has energy E, volume V and entropy S. In a homogeneous system these are usually "extensive" quantities (scaling proportional to the size of the system), which implies that $E(\lambda S, \lambda V) = \lambda E(S, V)$ for any $\lambda > 0$. Use this to show that if $E(S, V)$ is not convex then there exist S, V, S_1, S_2, V_1, V_2 such that $S_1 + S_2 = S$, $V_1 + V_2 = V$ and $E(S_1, V_1) + E(S_2, V_2) < E(S, V)$. Recall that E increases (unboundedly) with S so there exists $S_1' > S_1$ such that $E(S_1', V_1) + E(S_2, V_2) = E(S, V)$. Deduce that the system can be divided into two parts such that the total energy and volume are unchanged but the total entropy has increased. (The second law of thermodynamics asserts that the entropy of an isolated system is an increasing function of time. Hence this argument suggests that a homogeneous system with non-convex $E(S, V)$ will be unstable against splitting into smaller parts and becoming inhomogeneous.)

5. The area A of a triangle with sides a, b, c is given by

$$A = \sqrt{[s(s-a)(s-b)(s-c)]}, \quad \text{where } s = \tfrac{1}{2}(a+b+c).$$

(i) Show that of all triangles of given perimeter $2s$, the triangle of largest area is equilateral.

(ii) Find (in terms of the perimeter) the largest possible area of a right-angled triangle of given perimeter.

6. Find the maximum volume of a cuboid inscribed inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

You may assume that the axes of the cuboid are aligned with those of the ellipsoid.

7. Let (θ, ϕ) be the standard angular coordinates on the unit sphere. A function $\phi(\theta)$ defines a path on this sphere. Given that θ increases by $\delta\theta$ over a short segment of this path, show that the length of the path segment is $\sqrt{1 + (\phi' \sin\theta)^2}\;\delta\theta$ to first order in $\delta\theta$. Hence find a functional $L[\phi]$ for the total length of a path between any two points on the unit sphere. Use your result to show that the paths of mimimal length are segments of great circles.

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8. A soap film is bounded by two circular wires at $r = a$, $z = \pm b$ in cylindrical polar coordinates (r, θ, z). Given that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is

$$r = c \cosh(z/c) \,,$$

where c satisfies the condition $a/c = \cosh(b/c)$. Show graphically that this condition has no solution for c if b/a is larger that a certain critical ratio. What happens to the soap surface as b/a is increased from below this ratio to above it?

9. It has been suggested that Crossrail, the new rail connection under London, should include a frictionless tunnel in which fuel-less trains can run under gravity. The trains are released from rest at the point of departure (Stratford) and are then allowed to run freely until arriving at their destination (Acton) at the same level. Assuming that the acceleration g due to gravity is uniform, show that the minimum travel time is $\sqrt{2\pi\ell/g}$, where ℓ is the horizontal distance between the departure and arrival points. Comment on the quality of the likely travel experience at the departure and arrival points.

10. In an optical medium filling the region $0 < y < h$, the speed of light is

$$c(y) = \frac{c_0}{\sqrt{1 - ky}} \,, \qquad 0 < k < 1/h \,.$$

Show that the paths of light rays in the medium are parabolic. Show also that if a ray enters the medium at $(-x_0, 0)$ and leaves it at $(x_0, 0)$ then

$$(kx_0)^2 = 4ky_0(1 - ky_0) \,,$$

where y_0 $(< h)$ is the greatest value of y attained on the ray path.

11. Determine all functions $u(x)$ that extremize the functional

$$I[u] = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} u'^2 + (1 - \cos u) \right\} dx \qquad (*)$$

subject to the boundary conditions

$$\lim_{x \to -\infty} u(x) = 0 \,, \qquad \lim_{x \to \infty} u(x) = 2\pi \,.$$

For $u(x)$ satisfying these boundary conditions, show that

$$I[u] = 8 + \frac{1}{2} \int_{-\infty}^{\infty} \left(u' - 2 \sin(u/2) \right)^2 dx \,.$$

Deduce that $I[u] \geq 8$ for such functions, and write down a *first-order* differential equation that u must satisfy in order to realise this lower bound. How is this differential equation related to the second-order Euler-Lagrange equation for the function $I[u]$ of $(*)$?

12. *Dido's problem.* An area A of a field is enclosed by a length ℓ of flexible fencing with its ends attached a distance a apart on a straight wall, where $a < \ell < \frac{1}{2}\pi a$. Show that A is maximised, for fixed ℓ, by the arc of a circle, and derive an equation relating the radius of the circle to the location of its centre. Comment on the case of $\ell > \frac{1}{2}\pi a$.

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