

Complex Methods: Example Sheet 2

Part IB, Lent Term 2022

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for $\log(1+z)$, etc.

$$(i) z/\log(1+z) \quad (ii) (\cos z)^{1/2} - 1 \quad (iii) \log(1+e^z) \quad (iv) e^{e^z}$$

State the range of values of z for which each series converges.

How would your answers differ if you assumed branches different from the principal branch?

2. Let a, b be complex constants, $0 < |a| < |b|$. Use partial fractions to find the Laurent expansions of $1/\{(z-a)(z-b)\}$ about $z=0$ in each of the regions $|z| < |a|$, $|a| < |z| < |b|$ and $|z| > |b|$.

3. Find the first three terms of the Laurent expansion of $f(z) = \frac{1}{\sin^2 z}$ valid for $0 < |z| < \pi$.

* Show that the function $g(z) = f(z) - z^{-2} - (z+\pi)^{-2} - (z-\pi)^{-2}$ has only removable singularities in $|z| < 2\pi$. Explain how to remove them to obtain a function $G(z)$ analytic in that region. Find a Taylor Series for $G(z)$ about the origin and explain why it must be convergent in $|z| < 2\pi$. Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of $f(z)$ valid for $\pi < |z| < 2\pi$.

4. Show that if $f(z)$ has a zero of order M and $g(z)$ a zero of order N at $z = z_0$, then $f(z)/g(z)$ has a zero of order $M - N$ if $M > N$, a removable singularity if $M = N$, and a pole of order $N - M$ if $M < N$. Show also that $1/f(z)$ has a pole of order N if and only if $f(z)$ has a zero of order N .

5. Write down the location and type of each of the singularities of the following functions:

$$(i) \frac{1}{z^3(z-1)^2} \quad (ii) \tan z \quad (iii) z \coth z \quad (iv) \frac{e^z - e}{(1-z)^3}$$
$$(v) \exp(\tan z) \quad (vi) \sinh \frac{z}{z^2 - 1} \quad (vii) \log(1 + e^z) \quad (viii) \tan(z^{-1})$$

Integration and residues

6. Evaluate $\int z dz$ along the straight line from -1 to $+1$, and along the semicircular contour in the upper half-plane between the same two points; and evaluate $\oint_{\gamma} \bar{z} dz$ when γ is the circle $|z| = 1$, and when γ is the circle $|z - 1| = 1$.

7. (i) Show that if $f(z)$ and $g(z)$ are analytic, and g has a simple zero at $z = z_0$, the residue of $f(z)/g(z)$ at $z = z_0$ is $f(z_0)/g'(z_0)$. In particular, show that $f(z)/(z - z_0)$ has residue $f(z_0)$.
(ii) Prove the formula for the residue of a function $f(z)$ that has a pole of order N at $z = z_0$:

$$\lim_{z \rightarrow z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} ((z - z_0)^N f(z)) \right\}.$$

- (iii) Find the residues of the poles in question 5.

8. Evaluate, using Cauchy's theorem or the residue theorem,

$$(i) \oint_{\gamma_1} \frac{dz}{1+z^2} \quad (ii) \oint_{\gamma_2} \frac{dz}{1+z^2} \quad (iii) \oint_{\gamma_3} \frac{e^z \cot z \, dz}{1+z^2} \quad * (iv) \oint_{\gamma_4} \frac{z^3 e^{1/z} \, dz}{1+z}$$

where $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$, γ_1 is the elliptical contour $x^2 + 4y^2 = 1$, γ_2 is the circle $|z| = \sqrt{2}$, γ_3 is the circle $|z - (2 + i)| = \sqrt{2}$ and γ_4 is the circle $|z| = 2$, all traversed anti-clockwise.

9. By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle and applying the residue theorem, evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a \cos \theta + a^2} \, d\theta$$

where a is real, $a > 1$, and n is a non-negative integer.

* Obtain the same result using Cauchy's integral formula instead of the residue theorem.

The calculus of residues

10. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x+x^2}$.

11. By integrating around a keyhole contour, show that

$$\int_0^{\infty} \frac{x^{a-1} \, dx}{1+x} = \frac{\pi}{\sin \pi a} \quad (0 < a < 1).$$

Explain why the given restrictions on the value of a are necessary.

* 12. By integrating around a contour involving the real axis and the line $z = re^{2\pi i/n}$, evaluate $\int_0^{\infty} dx/(1+x^n)$, $n \geq 2$. Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

$$(i) \int_0^{\infty} \frac{\cos x}{(1+x^2)^3} \, dx = \frac{7\pi}{16e} \quad (ii) \int_0^{\infty} \frac{x^2}{\cosh x} \, dx = \frac{\pi^3}{8}$$

$$(iii) \int_0^{\infty} \frac{\log x}{1+x^2} \, dx = 0 \quad * (iv) \int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}$$

[For part (ii), use a rectangular contour. For part (iii), integrate $(\log z)^2/(1+z^2)$ around a keyhole, or $(\log z)/(1+z^2)$ along the real axis (or both). What goes wrong with $(\log z)/(1+z^2)$ around a keyhole?]

* 14. Let $P(z)$ be a non-constant polynomial. Consider the contour integral $I = \oint_{\gamma} (P'(z)/P(z)) \, dz$. Show that, if γ is a contour that encloses no zeros of P , then $I = 0$.

Evaluate the limit of I as $R \rightarrow \infty$, where γ is the circle $|z| = R$, and deduce that P has at least one zero in the complex plane.

15. By considering the integral of $(\cot z)/(z^2 + \pi^2 a^2)$ around a suitable large contour, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$$

provided that ia is not an integer. By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

Find an expression for $\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$ and take the limit as $a \rightarrow 0$ to deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.