Complex Methods: Example Sheet 2

Part IB, Lent Term 2021
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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for $\log(1+z)$, etc.

(i) $z/\log(1+z)$ (ii) $(\cos z)^{1/2}-1$ (iii) $\log(1+e^z)$ (iv) e^{e^z}

State the range of values of z for which each series converges.

How would your answers differ if you assumed branches different from the principal branch?

2. Let a, b be complex constants, $0<|a|<|b|$. Use partial fractions to find the Laurent expansions of $1/\{(z-a)(z-b)\}$ about $z=0$ in each of the regions $|z|<|a|$, $|a|<|z|<|b|$ and $|z|>|b|$.

3. Find the first three terms of the Laurent expansion of $f(z)=\frac{1}{\sin^2 z}$ valid for $0<|z|<\pi$.

$*$ Show that the function $g(z)=f(z)-z^{-2}-(z+\pi)^{-2}-(z-\pi)^{-2}$ has only removable singularities in $|z|<2\pi$. Explain how to remove them to obtain a function $G(z)$ analytic in that region. Find a Taylor Series for $G(z)$ about the origin and explain why it must be convergent in $|z|<2\pi$. Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of $f(z)$ valid for $\pi<|z|<2\pi$.

4. Show that if $f(z)$ has a zero of order M and $g(z)$ a zero of order N at $z=z_0$, then $f(z)/g(z)$ has a zero of order $M-N$ if $M>N$, a removable singularity if $M=N$, and a pole of order $N-M$ if $M<N$. Show also that $1/f(z)$ has a pole of order N if and only if $f(z)$ has a zero of order N.

5. Write down the location and type of each of the singularities of the following functions:

(i) $\dfrac{1}{z^3(z-1)^2}$ (ii) $\tan z$ (iii) $z\coth z$ (iv) $\dfrac{e^z-e}{(1-z)^3}$

(v) $\exp(\tan z)$ (vi) $\sinh\dfrac{z}{z^2-1}$ (vii) $\log(1+e^z)$ (viii) $\tan(z^{-1})$

Integration and residues

6. Evaluate $\int z\,\mathrm{d}z$ along the straight line from -1 to $+1$, and along the semicircular contour in the upper half-plane between the same two points; and evaluate $\oint_\gamma \bar{z}\,\mathrm{d}z$ when γ is the circle $|z|=1$, and when γ is the circle $|z-1|=1$.

7. (i) Show that if $f(z)$ and $g(z)$ are analytic, and g has a simple zero at $z=z_0$, the residue of $f(z)/g(z)$ at $z=z_0$ is $f(z_0)/g'(z_0)$. In particular, show that $f(z)/(z-z_0)$ has residue $f(z_0)$.

(ii) Prove the formula for the residue of a function $f(z)$ that has a pole of order N at $z=z_0$:

$$\lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\big((z-z_0)^N f(z)\big)\right\}.$$

(iii) Find the residues of the poles in question 5.

8. Evaluate, using Cauchy's theorem or the residue theorem,

$$\text{(i)} \oint_{\gamma_1} \frac{\mathrm{d}z}{1+z^2} \qquad \text{(ii)} \oint_{\gamma_2} \frac{\mathrm{d}z}{1+z^2} \qquad \text{(iii)} \oint_{\gamma_3} \frac{e^z \cot z \,\mathrm{d}z}{1+z^2} \qquad * \text{ (iv)} \oint_{\gamma_4} \frac{z^3 e^{1/z}\,\mathrm{d}z}{1+z}$$

where $x = \mathsf{Re}(z),\ y = \mathsf{Im}(z)$, γ_1 is the elliptical contour $x^2 + 4y^2 = 1$, γ_2 is the circle $|z| = \sqrt{2}$, γ_3 is the circle $|z - (2 + \mathrm{i})| = \sqrt{2}$ and γ_4 is the circle $|z| = 2$, all traversed anti-clockwise.

9. By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle and applying the residue theorem, evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta$$

where a is real, $a > 1$, and n is a non-negative integer.

$*$ Obtain the same result using Cauchy's integral formula instead of the residue theorem.

The calculus of residues

10. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1 + x + x^2}$.

11. By integrating around a keyhole contour, show that

$$\int_0^{\infty} \frac{x^{a-1}\,\mathrm{d}x}{1+x} = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1).$$

Explain why the given restrictions on the value of a are necessary.

$*$ **12.** By integrating around a contour involving the real axis and the line $z = re^{2\pi\mathrm{i}/n}$, evaluate $\int_0^{\infty} \mathrm{d}x/(1+x^n)$, $n \geq 2$. Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

$$\text{(i)} \int_0^{\infty} \frac{\cos x}{(1+x^2)^3}\,\mathrm{d}x = \frac{7\pi}{16e} \qquad \text{(ii)} \int_0^{\infty} \frac{x^2}{\cosh x}\,\mathrm{d}x = \frac{\pi^3}{8}$$

$$\text{(iii)} \int_0^{\infty} \frac{\log x}{1+x^2}\,\mathrm{d}x = 0 \qquad * \text{ (iv)} \int_0^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}$$

[*For part (ii), use a rectangular contour. For part (iii), integrate* $(\log z)^2/(1+z^2)$ *around a keyhole, or* $(\log z)/(1+z^2)$ *along the real axis (or both). What goes wrong with* $(\log z)/(1+z^2)$ *around a keyhole?*]

$*$ **14.** Let $P(z)$ be a non-constant polynomial. Consider the contour integral $I = \oint_\gamma \big(P'(z)/P(z)\big)\,\mathrm{d}z$. Show that, if γ is a contour that encloses no zeros of P, then $I = 0$.

Evaluate the limit of I as $R \to \infty$, where γ is the circle $|z| = R$, and deduce that P has at least one zero in the complex plane.

15. By considering the integral of $(\cot z)/(z^2 + \pi^2 a^2)$ around a suitable large contour, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a}\coth \pi a$$

provided that $\mathrm{i}a$ is not an integer. By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

Find an expression for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$ and take the limit as $a \to 0$ to deduce the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.