Mathematical Tripos Part IB Michaelmas 2023
Methods, Example Sheet 1 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Recall from lectures $c_n(\theta) = \cos(2\pi n\theta/L)$ and $s_n(\theta) = \sin(2\pi n\theta/L)$. Verify the orthogonality relations

$$\langle c_n, c_m\rangle = \langle s_n, s_m\rangle = \tfrac{1}{2}L\delta_{mn}, \quad \langle 1, c_n\rangle = \langle 1, s_m\rangle = \langle c_n, s_m\rangle = 0 \quad m,n \geq 1$$

where $\langle f, g\rangle = \int_0^L f(\theta)\overline{g(\theta)}\,\mathrm{d}\theta$. This confirms $\{1, c_n, s_n\}_{n=1}^{\infty}$ are orthogonal, as stated in lectures.

2. Consider 2-periodic function $f : \mathbf{R} \to \mathbf{R}$ with $f(\theta) = \left(1-\theta^2\right)^2$ when $\theta \in [-1,1)$. Show that it has Fourier series

$$f(\theta) \sim \frac{8}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos(n\pi\theta).$$

Can we replace '$\sim$' with '$=$' in this case? Sketch the graph of f and comment on the number of continuous derivatives it has and the relation to the decay of the Fourier coefficients.

3. Suppose $f(\theta) = \theta^2$ when $\theta \in [0,\pi)$.
(i) Construct (a) the Sine series for f and (b) the Cosine series for f, each having period 2π. Sketch the 2π-periodic functions obtained in (a) and (b) in the range $\theta \in [-6\pi, 6\pi)$.
(ii) If the series in (a) and (b) are formally differentiated term-by-term, are the resulting series related to the Fourier series for 2π-periodic functions $g, h : \mathbf{R} \to \mathbf{R}$ for which $g(\theta) = 2\theta$ and $h(\theta) = 2|\theta|$ when $\theta \in [-\pi, \pi)$?

4. Find the complex Fourier series for the 2π-periodic function $f : \mathbf{R} \to \mathbf{R}$ for which $f(\theta) = e^{\theta}$ when $\theta \in [-\pi,\pi)$. Using Parseval's theorem, deduce that

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{1}{2}\left(\pi\coth\pi - 1\right).$$

Obtain the same result by evaluating the complex Fourier series at an appropriate point in $[-\pi,\pi)$.

5. We say a sequence $\{r_n\}_{n\in\mathbf{Z}}$ *decays rapidly* if $|n|^k r_n \to 0$ as $|n| \to \infty$ for every $k \geq 0$.
(i) Let f be a smooth, L-periodic function. Show that the complex Fourier coefficients $\{\hat{f}_n\}$ decay rapidly.
(ii) Construct an L-periodic function with rapidly decaying, non-zero complex Fourier coefficients.

6. By considering the Sturm-Liouville problem for $y = y(x)$

$$\begin{cases} y'' + \lambda y = 0, \quad 0 < x < L, \\ y'(0) = 0, \\ y'(L) = 0, \end{cases}$$

re-derive the Cosine series representation for any $f \in C^2[0,L]$ with $f'(0) = f'(L) = 0$.

7. Prove that the boundary value problem for $y = y(x)$

$$\begin{cases} y'' + \lambda y = 0, \quad 0 < x < 1, \\ y(0) = 0, \\ y(1) + y'(1) = 0. \end{cases}$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and indicate roughly the behaviour of λ_n as $n \to \infty$.

8. Express the following eigenvalue problems as Sturm-Liouville problems on $[-1,1]$ and $[0,1]$, respectively:

$$\text{(i)} \ \left(1-x^2\right)y'' - 2xy' + \lambda y = 0, \qquad \text{(ii)} \ x(1-x)y'' - (ax-b)y' + \lambda y = 0,$$

where $a > b > 0$ are constant and λ is constant. Are either of these problems singular?
(iii) Find the eigenvalues and eigenfunctions of the boundary value problem for $y = y(x)$

$$\begin{cases} y'' + 4y' + (4+\lambda)y = 0, \quad 0 < x < 1, \\ y(0) = 0, \\ y(1) = 0. \end{cases}$$

What is the orthogonality relation for these eigenfunctions?

9. Define the functions $q_n(x) = \frac{1}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^2-1)^n$ for $n = 1, 2, \ldots$

(a) Show (i) q_n is a polynomial of degree n;
(ii) $q_n(1) = 1$ for all n;
(iii) q_n satisfies Legendre's equation.

(b) Deduce (i) $q_n = P_n$;
(ii) $\int_{-1}^{1} P_n(x)^2 \,\mathrm{d}x = 2/(2n+1)$;
(iii) $\int_{-1}^{1} x^m P_n(x)\,\mathrm{d}x = 0$ if $m < n$.

Hint: for (a)(iii) show $u_n = (x^2-1)^n$ satisfies $(x^2-1)u_n' - 2nxu_n = 0$ and differentiate further.

10. Recall from lectures that if $y_\alpha(r) = J_m(\alpha r)$, where J_m is the mth order Bessel function of the 1st kind, then

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(r\frac{\mathrm{d}y_\alpha}{\mathrm{d}r}\right) + \frac{m^2}{r}y_\alpha = \alpha^2 r y_\alpha, \quad r \in (0,1).$$

Show that if $y_\beta(r) = J_m(\beta r)$ then $[r(y_\alpha y_\beta' - y_\beta y_\alpha')]' = \left(\alpha^2 - \beta^2\right) r y_\alpha y_\beta$. Deduce that

$$\int_0^1 J_m(\alpha r) J_m(\beta r) r \,\mathrm{d}r = \frac{\beta J_m(\alpha) J_m'(\beta) - \alpha J_m(\beta) J_m'(\alpha)}{\alpha^2 - \beta^2}, \quad \alpha \neq \beta.$$

Use this result to show that $\int_0^1 J_m(j_{mk} r) J_m(j_{m\ell} r) r\,\mathrm{d}r = \frac{1}{2}[J_m'(j_{mk})]^2 \delta_{k\ell}$, where $J_m(j_{mk}) = 0$, $k = 1, 2, \ldots$

Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

11. Let f be the 2π-periodic square wave for which $f(\theta) = 1$ on $[0,\pi)$ and $f(\theta) = 0$ on $[\pi, 2\pi)$.
(i) Sketch the graph of f and show that

$$f(\theta) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\theta]}{2n-1}.$$

(ii) Let $S_N f$ denote the partial Fourier series for f. By considering $\sum_{n=1}^{N} \cos[(2n-1)\theta]$, or otherwise, show

$$(S_N f)(\theta) = \frac{1}{2} + \frac{1}{\pi}\int_0^\theta \frac{\sin(2N\phi)}{\sin\phi}\,\mathrm{d}\phi.$$

(iii) Deduce that $(S_N f)(\theta)$ has a local extrema at $\theta = \pi m/2N$, $m \in \mathbf{Z} \setminus 2N\mathbf{Z}$ and that for large N

$$(S_N f)\left(\frac{\pi}{2N}\right) \approx \frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin u}{u}\,\mathrm{d}u = \frac{1}{2} + \int_0^{\pi/2} \frac{\sin u}{u(\pi - u)}\,\mathrm{d}u \geq 1.08.$$

Hint for lower bound: $\sin u \geq u - u^3/3!$. Comment on the accuracy of partial Fourier series at discontinuities.

12. Set $V = \{y \in C^2[a,b] : y(a) = y(b) = 0\}$ (i.e. Dirichlet boundary conditions) and let $L = \frac{1}{w}\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}}{\mathrm{d}x}\right) + q\right]$ be a Sturm-Liouville operator with p, q, w smooth and $p, w > 0$ on $[a,b]$. Consider the *Rayleigh quotient*

$$R[y] = \frac{\int_a^b \left[p\,(y')^2 + qy^2\right]\mathrm{d}x}{\int_a^b wy^2\,\mathrm{d}x}, \quad y \in V.$$

(a) By considering $\langle Ly, y\rangle_w$, show that if $y \in V$ satisfies $Ly = \lambda y$ then $\lambda = R[y]$.
(b) Let $\lambda_1 = \inf_{y \in V\setminus\{0\}} R[y]$ and suppose that there exists a $y_1 \in V$ such that $R[y_1] = \lambda_1$. If we set

$$F(\epsilon) = R[y_1 + \epsilon\eta],$$

where $\eta \in V$, explain why $F'(0) = 0$. Hence show that[1] $Ly_1 = \lambda_1 y_1$. Comment on this result in relation to finding the smallest eigenvalue of L. How might you try to find the second smallest? (*Hint: orthogonality*).
(c) Take $[a,b] = [0,1]$ and $L = -\mathrm{d}^2/\mathrm{d}x^2$. Compute $R[y]$ where $y(x) = x(1-x)$ and deduce $\lambda_{\min} = \pi^2 \leq 10$.

[1] You may assume that if $\int_a^b f\eta\,\mathrm{d}x = 0$ for all $\eta \in V$ then $f = 0$.