

METHODS — EXAMPLES I

Fourier series

1. *Fourier coefficients (full-range series).* For the periodic function $f(x) = (x^2 - 1)^2$ on the interval $-1 \leq x < 1$, show that it has the Fourier series

$$f(x) = \frac{8}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi x.$$

Sketch the function $f(x)$ and comment on its differentiability and the order of the terms in its Fourier series as $n \rightarrow \infty$.

2. *Fourier coefficients (half-range series).* Suppose that $f(x) = x^2$ for $0 \leq x < \pi$. Express $f(x)$ as (a) a Fourier sine series, and (b) a cosine series, each having period 2π . Sketch the functions represented by (a) and (b) in the range -6π to 6π . If the series (a) and (b) are differentiated term-by-term, how are the answers related (if at all) to the Fourier series for $g(x) = 2x$ and $h(x) = 2|x|$ each in the range $[-\pi, \pi]$?

3. *Series summation.* Find the Fourier series of the 2π -periodic function $f(x) = e^x$ on $[-\pi, \pi]$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1) \quad .$$

4. *Parseval's identity and a low pass filter.* (i) Given that a function $f(t)$ defined over the interval $(-T, T)$ has the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right], \quad \text{show that} \quad \frac{1}{T} \int_{-T}^T [f(t)]^2 dt = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

where you may assume $f(t)$ is such that this series is convergent.

(ii) A unit amplitude square wave of period $2T$ is given by $f(t) = 1$ for $0 < t < T$ and $f(t) = -1$, for $-T < t < 0$. Suppose this is the input for a system which permits angular frequencies less than $\frac{9}{2}\pi T^{-1}$ to be perfectly transmitted and frequencies greater than $\frac{9}{2}\pi T^{-1}$ to be perfectly absorbed. Calculate the form of the output. The power is proportional to the mean value of $f^2(t)$; what fraction of the incident power is transmitted?

5. *Discontinuities and the Wilbraham-Gibbs phenomenon*.* (i) Suppose that f is a 2π -periodic square wave given by

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi. \end{cases} \quad \text{Sketch } f \text{ and show that its Fourier series is } \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

(ii) Now define the partial sum of this series as
$$S_N(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^N \frac{\sin(2n-1)x}{2n-1} \quad ,$$

and find the following expression
$$S_N(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^x \frac{\sin 2Nt}{\sin t} dt \quad . \quad \text{[Hint: consider } \sum_{n=1}^N \cos(2n-1)x \text{.]}$$

(iii) Deduce that $S_N(x)$ has extrema at $x = m\pi/2N$, $m = 1, 2, \dots, 2N - 1, 2N + 1, \dots$, (i.e. all integer m except $m = 2kN$, k integer) and that the height of the first maximum for large N is approximately

$$S_N\left(\frac{\pi}{2N}\right) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin u du}{u} (\simeq 1.089).$$

Comment on the accuracy of Fourier series at discontinuities.

[This question takes you through some important steps which are used in the proof of Fourier's theorem - refer, for example, to chapter 14 of *Jeffreys & Jeffreys*. Henry Wilbraham was a fellow of Trinity who wrote a paper (when he was 22) deriving this result 50 years before Gibbs, after whom it is commonly named.]

Sturm-Liouville theory

6. Eigenfunctions and eigenvalues. Prove that the boundary value problem

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) + y'(1) = 0,$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \dots$. Indicate roughly the behaviour of λ_n as $n \rightarrow \infty$.

7. Recasting in Sturm-Liouville form. Express the following equations in Sturm-Liouville form:

$$(i) \quad (1 - x^2)y'' - 2xy' + \lambda y = 0, \quad (ii) \quad x(x-1)y'' + [ax - b]y' + \lambda y = 0,$$

where $a > b$ and λ are constants.

(iii) Find the eigenvalues and eigenfunctions for

$$y'' + 4y' + (4 + \lambda)y = 0, \quad y(0) = y(1) = 0.$$

What is the orthogonality relation for these eigenfunctions?

8. Bessel's equation. (i) Show that the eigenvalues of the Sturm-Liouville problem

$$\frac{d}{dx} \left(x \frac{du}{dx} \right) + \lambda x u = 0, \quad 0 < x < 1,$$

with $u(x)$ bounded as $x \rightarrow 0$ and $u(1) = 0$, are $\lambda = j_n^2$ ($n = 1, 2, \dots$), where the j_n are the zeros of the Bessel function $J_0(z)$, arranged in ascending order. [Note: Bessel's equation of order zero is $\frac{d}{dz} \left(z \frac{dy}{dz} \right) + zy = 0$, ($z > 0$), which you may assume has one solution $J_0(z)$ defined as

$$J_0(z) = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{2^{2m} (m!)^2}$$

and a second solution that is the sum of a regular function and $J_0(z) \log z$.]

(ii) Using integration by parts on the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$, show that

$$\int_0^1 J_0(\alpha x) J_0(\beta x) x dx = \frac{\beta J_0(\alpha) J_0'(\beta) - \alpha J_0(\beta) J_0'(\alpha)}{\alpha^2 - \beta^2} \quad (\beta \neq \alpha)$$

$$\int_0^1 J_0(j_n x) J_0(j_m x) x dx = 0, \quad (n \neq m), \quad \int_0^1 [J_0(j_n x)]^2 x dx = \frac{1}{2} [J_0'(j_n)]^2. \quad [\text{Hint: Consider } \beta = j_n + \epsilon \text{ as } \epsilon \rightarrow 0.]$$

(iii) Assume that the inhomogeneous equation

$$\frac{d}{dx} \left(x \frac{du}{dx} \right) + \tilde{\lambda} x u = x f(x),$$

where $\tilde{\lambda}$ is not an eigenvalue, has a unique solution such that $u(x)$ is bounded as $x \rightarrow 0$ and $u(1) = 0$. Assuming also that $f(x)$ satisfies the same boundary conditions as u and the completeness of the eigenfunctions $J_0(j_n x)$, obtain the eigenfunction expansion of u .

9. Higher order self-adjoint form*. Show that the fourth-order differential operator

$$\mathcal{L} = \sum_{r=0}^4 p_r(x) \frac{d^r}{dx^r},$$

where the $p_r(x)$ are real functions, is self-adjoint (for appropriate choices of boundary conditions, which you do not need to determine) only if $p_3 = 2p_4$, $p_1 = p_2 - p_4''$.

Considering a specific example, show that the boundary value problem

$$-y'''' + \lambda y = 0; \quad y(0) = y(1) = y'(0) = y'(1) = 0,$$

where $\lambda > 0$ is real, has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \dots$. Indicate roughly the behaviour of λ_n as $n \rightarrow \infty$.

*If you find any errors in the Methods Examples sheets, please inform your supervisor or email epss@damtp.cam.ac.uk.