Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Using the method of characteristics, find the solution to each of the initial value problems:

$$\text{(i) } u_x + yu_y = 0,\ u(0,y) = y^3; \qquad \text{(ii) } u_x + u_y + u = e^{x+2y},\ u(x,0) = 0.$$

2. Tricomi's equation in $\mathbf{R}^2$ is $u_{xx} + xu_{yy} = 0$.
(i) Determine the regions in $\mathbf{R}^2$ where Tricomi's equation is (a) elliptic; (b) parabolic; (c) hyperbolic.
(ii) For the hyperbolic region, determine the characteristic curves. Hence put Tricomi's equation in canonical form.

3. Reduce the equation $u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$ to the canonical form $U_{\eta\xi} = 0$ in the hyperbolic region. Deduce that the general solution to the original equation is $u(x,y) = f(x + 2\sqrt{-y}) + g(x - 2\sqrt{-y})$ for arbitrary functions f, g.

4. Let $F, G : \mathbf{R}^n \to \mathbf{C}$ be smooth functions that decay rapidly as $|\mathbf{x}| \to \infty$. Using the identity

$$\delta(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi)^n} \int e^{\mathrm{i}\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})}\, \mathrm{d}^n\boldsymbol{\lambda} \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$$

establish Parseval's theorem for the Fourier transform on $\mathbf{R}^n$

$$\frac{1}{(2\pi)^n} \int \hat{F}(\boldsymbol{\lambda})\overline{\hat{G}(\boldsymbol{\lambda})}\, \mathrm{d}^n\boldsymbol{\lambda} = \int F(\mathbf{x})\overline{G(\mathbf{x})}\, \mathrm{d}^n\mathbf{x} \quad \text{hence} \quad \frac{1}{(2\pi)^n} \int |\hat{F}(\boldsymbol{\lambda})|^2\, \mathrm{d}^n\boldsymbol{\lambda} = \int |F(\mathbf{x})|^2\, \mathrm{d}^n\mathbf{x}.$$

5. Consider the initial value problem for the heat equation on $\mathbf{R}^n$

$$(\dagger) \quad \begin{cases} u_t - \kappa\Delta u = F(\mathbf{x}, t), & (\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty) \\ u(\mathbf{x}, 0) = f(\mathbf{x}), & \mathbf{x} \in \mathbf{R}^n \end{cases}$$

Let u_i denote the solution to (†) that has initial data f_i. Using the Fourier transform and Parseval's theorem, show

$$\|u_1(\cdot, t) - u_2(\cdot, t)\| \le C\|f_1 - f_2\| \quad \text{where} \quad \|u(\cdot, t)\|^2 = \int |u(\mathbf{x}, t)|^2\, \mathrm{d}^n\mathbf{x} \quad \text{etc.}$$

for some constant $C > 0$ you should determine. Deduce that (†) is well-posed with respect to the $\|\cdot\|$ norm.

6. Show that the Heat Kernel satisfies the semi-group property: $K_{t+s}(\mathbf{x}) = (K_t * K_s)(\mathbf{x})$.

7. Suppose $\mathcal{G} = \mathcal{G}(\mathbf{x}; \mathbf{y})$ is the Dirichlet Green's function for the Laplacian on a domain $\Omega \subset \mathbf{R}^n$, i.e. for each $\mathbf{y} \in \Omega$

$$\begin{cases} \Delta\mathcal{G} = \delta(\mathbf{x} - \mathbf{y}), & \mathbf{x} \in \Omega \\ \mathcal{G}(\mathbf{x}; \mathbf{y}) = 0, & \mathbf{x} \in \partial\Omega \end{cases}$$

Using Green's second identity, show that if $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$ then for $\mathbf{y} \in \Omega$

$$u(\mathbf{y}) = \int_{\partial\Omega} f(\mathbf{x}) \frac{\partial\mathcal{G}}{\partial\mathbf{n}}(\mathbf{x}; \mathbf{y})\, \mathrm{d}S(\mathbf{x}). \qquad (\ddagger)$$

8. Let $\Omega = \{(x, y) \in \mathbf{R}^2 : y > 0\}$ and consider the boundary value problem

$$\begin{cases} \Delta u = 0, & (x, y) \in \Omega, \\ u(x, 0) = f(x), & x \in \mathbf{R}, \\ u(x, y) \to 0, & \text{rapidly as } |x| + |y| \to \infty. \end{cases}$$

(a) Use the method of images to construct the Dirichlet Green's function for this problem and use (‡) to show

$$u(x, y) = \frac{y}{\pi} \int \frac{f(\xi)}{(x - \xi)^2 + y^2}\, \mathrm{d}\xi.$$

(b) Obtain the same result by first taking the Fourier transform (with respect to x) of $\Delta u = 0$ and $u(x, 0) = f(x)$.

9. Find the Dirichlet Green's function for the Laplacian on the unit ball $\Omega = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| \le 1\}$. *Hint: only one external charge is needed. Guess which line this external charge should lie on and go from there.*

10. An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts initial velocity $V\delta(x - x_0)$, where V is constant. Derive the position of the string for $t > 0$.

11. A semi-infinite string, fixed for all time at zero at $x = 0$ and at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is constant and $x_0 > 0$. Derive the position of the string for $t > 0$ and compare the solution to the infinite case in the previous question.

Additional problems

These questions should ***not*** *be attempted at the expense of earlier ones.*

12. Give a construction for the Dirichlet Green's function for the Laplacian on $\Omega = \{\mathbf{x} \in \mathbf{R}^n : x_1 > 0, \ldots, x_n > 0\}$.

13. Suppose that $u_{tt} - c^2\Delta u = 0$ on $\mathbf{R}^n \times (0, \infty)$. Fix $(\mathbf{x}_0, t_0)$ and suppose that $u(\mathbf{x}, 0) = u_t(\mathbf{x}, 0) = 0$ for $\mathbf{x}$ in the ball $B_0 = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \le ct_0\}$. By considering the *energy* in the ball $B_t = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \le c(t_0 - t)\}$,

$$E(t) = \frac{1}{2}\int_{B_t} \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \right] \mathrm{d}^n \mathbf{x}$$

show that $u(\mathbf{x}, t) = 0$ inside the *backward light cone* $\Sigma_{t_0}(\mathbf{x}_0) = \{(\mathbf{x}, t) : 0 \le t \le t_0 \text{ and } |\mathbf{x} - \mathbf{x}_0| \le c(t_0 - t)\}$. This shows that the solution to the wave equation at $(\mathbf{x}_0, t_0)$ depends only on the initial data in the region B_0.

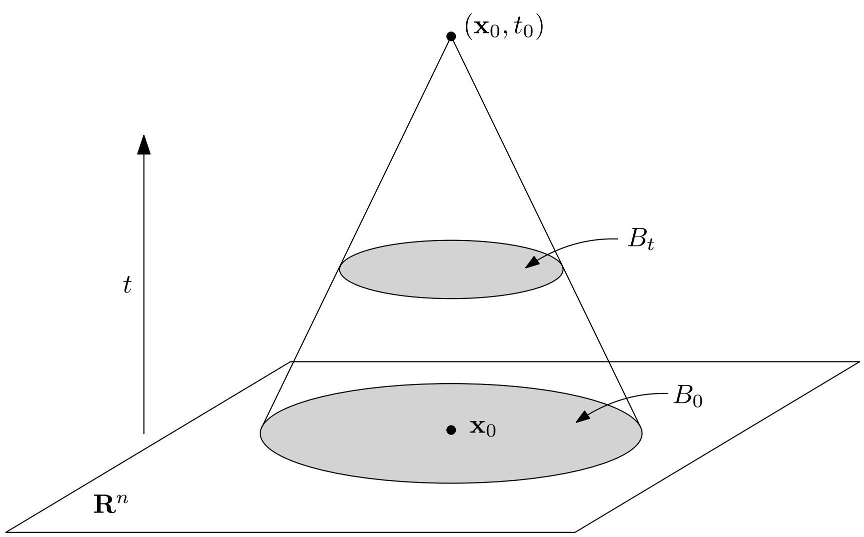

Figure 1: Backwards lightcone $\Sigma_{t_0}(\mathbf{x}_0)$.

Comment on the uniqueness of the solution to the initial value problem for the wave equation on $\mathbf{R}^n$.