

Example Sheet 2

1. A particle of mass  $m$  is confined to a one-dimensional box  $0 \leq x \leq a$  (the potential  $U(x)$  is zero inside the box, and infinite outside). Show that the energy eigenvalues are  $E_n = \hbar^2 \pi^2 n^2 / 2ma^2$  for  $n = 1, 2, \dots$ , and determine corresponding normalised energy eigenstates  $\chi_n(x)$ . Show that the expectation value and the uncertainty for a measurement of  $\hat{x}$  in the state  $\chi_n$  are given by

$$\langle \hat{x} \rangle_n = \frac{a}{2} \quad \text{and} \quad (\Delta x)_n^2 = \frac{a^2}{12} \left( 1 - \frac{6}{\pi^2 n^2} \right).$$

Does the limit  $n \rightarrow \infty$  agree with what you would expect for a classical particle in this potential?

2. Write down the time-independent Schrödinger equation for the wavefunction of a particle moving in a potential  $U(x) = -U\delta(x)$ , where  $U$  is a positive constant and  $\delta(x)$  is the Dirac delta function. Integrate the equation over the interval  $-\epsilon < x < \epsilon$ , for a positive constant  $\epsilon$ , and hence deduce that there is a discontinuity at  $x = 0$  in the derivative of  $\chi(x)$ :

$$\lim_{\epsilon \rightarrow 0} [\chi'(\epsilon) - \chi'(-\epsilon)] = -\frac{2mU}{\hbar^2} \chi(0).$$

By using this condition to relate appropriate solutions for  $x > 0$  and  $x < 0$ , find the unique bound and normalisable eigenstate of the Hamiltonian, and determine its energy eigenvalue  $E$  (with  $E < 0$ ).

3. Consider a square well potential with  $U(x) = -U$  for  $|x| < a$  and  $U(x) = 0$  otherwise ( $U$  is a positive constant). Show that there are no bound states (normalisable energy eigenfunctions) which satisfy  $\chi(-x) = -\chi(x)$  (i.e. which have odd parity) if  $a^2 U < (\pi\hbar)^2 / 8m$ .

4. Sketch the potential

$$U(x) = -\frac{\hbar^2}{m} \operatorname{sech}^2 x.$$

Show that the time-independent Schrödinger equation for a particle in this potential can be written

$$\hat{A}^\dagger \hat{A} \chi = (\mathcal{E} + 1) \chi$$

where  $\mathcal{E} = 2mE/\hbar^2$  and

$$\hat{A} = \frac{d}{dx} + \tanh x, \quad \hat{A}^\dagger = -\frac{d}{dx} + \tanh x.$$

Show, by integrating by parts, that for any normalised wavefunction  $\chi$ ,

$$\int_{-\infty}^{\infty} \chi^* \hat{A}^\dagger \hat{A} \chi dx = \int_{-\infty}^{\infty} (\hat{A} \chi)^* (\hat{A} \chi) dx$$

and deduce that the eigenvalues of  $\hat{A}^\dagger \hat{A}$  are non-negative. Hence show that the ground state (with lowest energy) has  $\mathcal{E} \geq -1$ . Show that a wavefunction  $\chi_0(x)$  is an energy eigenstate with  $\mathcal{E} = -1$  iff

$$\frac{d\chi_0}{dx} + \tanh x \chi_0 = 0.$$

Find and sketch  $\chi_0(x)$ .

5. Write down the Hamiltonian  $H$  for a harmonic oscillator of mass  $m$  and frequency  $\omega$ . Express  $\langle H \rangle$  in terms of  $\langle \hat{x} \rangle$ ,  $\langle \hat{p} \rangle$ ,  $\Delta x$  and  $\Delta p$ , all defined for some normalised state  $\psi$ . Use the Uncertainty Relation to deduce that  $E \geq \frac{1}{2} \hbar \omega$  for any energy eigenvalue  $E$ .

**6.** The energy levels of the harmonic oscillator are  $E_n = (n + \frac{1}{2})\hbar\omega$  for  $n = 0, 1, 2, \dots$  and the corresponding stationary state wavefunctions are

$$\chi_n(x) = h_n(y)e^{-y^2/2} \quad \text{where} \quad y = (m\omega/\hbar)^{1/2}x$$

and  $h_n$  is a polynomial of degree  $n$  with  $h_n(-y) = (-1)^n h_n(y)$ . Using *only* the orthogonality relations

$$(\chi_m, \chi_n) = \delta_{mn},$$

determine  $\chi_2$  and  $\chi_3$  up to an overall constant in each case.

Give an expression for the quantum state of the oscillator  $\psi(x, t)$  if the initial state is  $\psi(x, 0) = \sum_{n=0}^{\infty} c_n \chi_n(x)$ , where  $c_n$  are complex constants. Deduce that

$$|\psi(x, 2p\pi/\omega)|^2 = |\psi(-x, (2q+1)\pi/\omega)|^2$$

for any integers  $p, q \geq 0$ . Comment on this result, considering the particular case in which  $\psi(x, 0)$  is sharply peaked around position  $x = a$ .

**7.** A particle of mass  $m$  is in a one-dimensional infinite square well (a potential box) with  $U = 0$  for  $0 < x < a$  and  $U = \infty$  otherwise. The normalised wavefunction for the particle at time  $t = 0$  is

$$\psi(x, 0) = Cx(a - x).$$

(i) Determine the real constant  $C$ .

(ii) By expanding  $\psi(x, 0)$  as a linear combination of energy eigenfunctions (found in Example 1 above), obtain an expression for  $\psi(x, t)$ , the wavefunction at time  $t$ .

(iii) A measurement of the energy is made at time  $t > 0$ . Show that the probability that this yields the result  $E_n = \hbar^2 \pi^2 n^2 / 2ma^2$  is  $960/\pi^6 n^6$  if  $n$  is odd, and zero if  $n$  is even. Why should the result for  $n$  even be expected? Which value of the energy is most likely, and why is its probability so close to unity?

**8.** Consider the Schrödinger Equation in one dimension with potential  $U(x)$ . Show that for a stationary state, the probability current  $J$  is independent of  $x$ .

Now suppose that an energy eigenstate  $\chi(x)$  corresponds to scattering by the potential and that  $U(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Given the asymptotic behaviour

$$\chi(x) \sim e^{ikx} + Be^{-ikx} \quad (x \rightarrow -\infty) \quad \text{and} \quad \chi(x) \sim Ce^{ikx} \quad (x \rightarrow +\infty)$$

show that  $|B|^2 + |C|^2 = 1$ . How should this be interpreted?

**9.** A particle is incident on a potential barrier of width  $a$  and height  $U$ . Assuming that  $U = 2E$ , where  $E = \hbar^2 k^2 / 2m$  is the kinetic energy of the incident particle, find the transmission probability. [ *Work through the algebra, which simplifies in this case, rather than quoting the general result.* ]

**10.** Consider the time-independent Schrödinger Equation with potential  $U(x) = -U\delta(x)$ . Show that there is a scattering solution with energy eigenvalue  $E = \hbar^2 k^2 / 2m$  for any real  $k > 0$  and find the transmission and reflection coefficients  $A_{\text{tr}}(k)$  and  $A_{\text{ref}}(k)$  (that correspond to the transmission and reflection coefficients defined in the notes as  $T$  and  $R$  respectively). [ *Recall from Example 2 that the energy eigenfunction  $\chi$  is continuous, but satisfies  $\chi'(0+) - \chi'(0-) = -(2mU/\hbar^2)\chi(0)$ .* ]

Is the solution above still an eigenfunction of the Hamiltonian if  $k$  is allowed to take complex values? Show that  $A_{\text{tr}}(k)$  and  $A_{\text{ref}}(k)$  are singular at  $k = i\kappa$  for a certain real, positive value of  $\kappa$ . By first re-scaling the scattering solution, find a bound state (normalisable) solution in the potential. What is the energy of this bound state?